Our goal: For a function $f \in L^2_w(a,b)$ $\left(\|f\|_2 = \sqrt{\int_a^b (f(x))^2 w(x)\,dx} < \infty \right)$

find $p_n \in \mathbb{P}_n$ such that:

$$\|f - p_n\|_2^2 = \inf_{q \in \mathbb{P}_n} \|f - q\|_2^2$$

$$= \inf_{q \in \mathbb{P}_n} \int_a^b \left(f(x) - q(x)\right)^2 w(x)\,dx.$$

Ex: Let $n = 0$, $f(x) = -2x^2$ on $[-1,1]$, $w(x) = 1$.

Find $p_0 = c$ to minimize $\|f - p_0\|_2^2$.

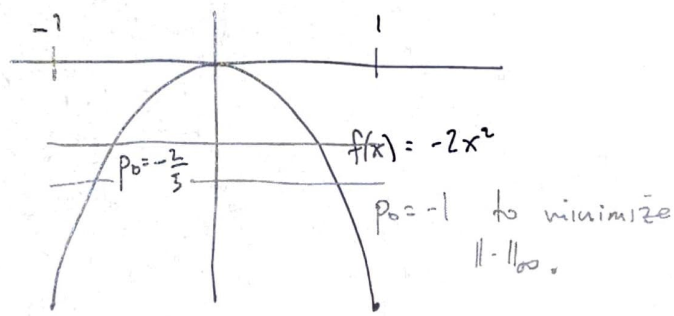
$$\|f - p_0\|_2^2 = \int_{-1}^{1} (-2x^2 - c)^2\,dx$$

$$= 2c^2 + \frac{8c}{3} + \frac{8}{5}$$

Find c to minimize this quantity.

$$\frac{d}{dc}\|f - p_0\|_2^2 = 4c + \frac{8}{5} = 0$$

$$\Rightarrow c = -\frac{2}{3}$$



What would the best approximation in $\|\cdot\|_\infty$ be?

Computing the best 2-norm approximation

Recall: The least squares solution to $A\vec{x} = \vec{b}$ is obtained by solving $A\vec{x} = \underbrace{QQ^T\vec{b}}$

projection of $\vec{b}$ onto col A, Q is obtained by applying the Gram-Schmidt process to the columns of A.

To apply Gram-Schmidt, all that was needed was a vector space and an inner product.

We can do <u>the</u> same thing for polynomial least squares:

① Trivially, P_n is a vector space.

② We can define an inner product on P_n by: $(f,g) = \int_a^b f(x)\, g(x)\, w(x)\, dx$.

Two functions are orthogonal if $(f,g) = 0$.

Let $p_0, p_1, p_2, \ldots, p_n$ be a basis for P_n.

The 2-norm approximation problem takes the form:

For $q(x) = \sum_{j=0}^{n} c_j p_j(x)$, $\|f-q\|_2^2$ is given by: (set $w=1$ for now)

$$A = \int_a^b \left(f(x) - \sum_{j=0}^{n} c_j p_j(x)\right)^2 dx$$

$$= \int_a^b (f(x))^2\, dx - 2\sum_{j=0}^{n} c_j \int_a^b f(x)\, p_j(x)\, dx + \sum_{j=0}^{n}\sum_{k=0}^{n} c_j c_k \int_a^b p_j(x)\, p_k(x)\, dx$$

$$= (f,f) - 2\sum_{j=0}^{n} c_j (f, p_j) + \sum_{j=0}^{n}\sum_{k=0}^{n} c_j c_k (p_j, p_k)$$

Writing down

$\nabla A = \vec{0}$, we have that:

$$\frac{\partial A}{\partial c_0} = 0, \quad \frac{\partial A}{\partial c_1} = 0, \quad \ldots, \quad \frac{\partial A}{\partial c_n} = 0$$

$$\Rightarrow \frac{\partial A}{\partial c_\ell} = -2(f, p_\ell) + 2\sum_{k=0}^{n} c_k (p_\ell, p_k) = 0$$

$$\Rightarrow \sum_{k=0}^{n} c_k (p_\ell, p_k) = (f, p_\ell)$$ ← $n+1$ linear equations, in $n+1$ variables $c_0 \ldots c_n$.

In matrix form:

$$\begin{pmatrix} (p_0,p_0) & \cdots & (p_0,p_n) \\ (p_1,p_0) & \cdots & (p_1,p_n) \\ \vdots & & \\ (p_n,p_0) & \cdots & (p_n,p_n) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} (f,p_0) \\ (f,p_1) \\ \vdots \\ (f,p_n) \end{pmatrix}$$

Therefore once this linear system has been solved, the best 2-norm approximation to f is:

$$q = c_0 p_0 + c_1 p_1 + \dots + c_n p_n.$$

If the p_ℓ's were orthonormal, i.e. $(p_\ell, p_k) = \delta_{\ell k}$,

then the above system is of the form:

$$I \vec{c} = RHS.$$

So $\Rightarrow$ $c_\ell = (f, p_\ell)$. (if p_ℓ's are orthonormal).