Function approximation

Polynomial interpolation mainly has application in function approximation, with respect to some norm:

For functions, some example norms are:

\[ \|f\|_\infty = \max_{x \in [a,b]} |f(x)| \]
\[ \|f\|_2 = \sqrt{\int_a^b |f(x)|^2 \, dx} \]  
\[ \|f\|_1 = \int_a^b |f(x)| \, dx \]  

Just like for \(n\)-dimensional vectors.

Norms of functions satisfy the same properties as those in the finite dimensional vector case:

1. \( \|f\| > 0 \), \( \|f\| = 0 \) iff \( f = 0 \)
2. \( \|cf\| = |c| \|f\| \)
3. \( \|f + g\| \leq \|f\| + \|g\| \)

Ex: The 2-norm of a function can be generalized by introducing a "weight" function \( w > 0 \):

\[ \|f\|_{2,w} = \sqrt{\int_a^b |f(x)|^2 w(x) \, dx} \]

So: the polynomial \( p_n \) of degree \( n \) that best approximates a function \( f \) in the \( \infty \)-norm is

\[ \min_{p_n} \|p_n - f\|_\infty \]  

Do not think of \( p_n \) as a polynomial interpolant of maximum pointwise error, \( f \).
From analysis class, we know that continuous functions $f$ on some finite interval can be approximated arbitrarily well by a polynomial of "some" degree; this result is known as the **Weierstrass Approximation Theorem**.

I.e. For any $\varepsilon > 0$, there exists a polynomial $p$ such that $\|f - p\|_{\infty} < \varepsilon$.

Unfortunately, this is a completely useless theorem for numerical approximation. It doesn't tell you how to find $p$!

The question of restricting $p \in \mathbb{P}_n$ is much more interesting, and actually useful.

To pose the problem:

For $n \geq 0$, find $p_n \in \mathbb{P}_n$ such that

$$\|f - p_n\|_{\infty} = \min_{q \in \mathbb{P}_n} \|f - q\|_{\infty}.$$  

**Theorem:** Such a $p_n$ exists, and is unique. (The proof does not tell us how to find $p_n$.)

In general, one cannot write down the **minimax polynomial**, i.e., the polynomial $p_n$ such that

$$\|f - p_n\|_{\infty} = \min_{q \in \mathbb{P}_n} \max_{x \in [a,b]} |f(x) - q(x)|$$
However, we can explicitly write down the minimax polynomial approximation to the monomial \( f(x) = x^{n+1} \) on \([0, 1]\):

\[
f(x) = x^{n+1}.
\]

**Theorem** Let \( n \geq 0 \), then \( \| p_n - f \|_\infty \), with \( f(x) = x^{n+1} \), is minimized when 
\[
p_n(x) = x^{n+1} - \frac{1}{2^n} \cos((n+1)\cos x),
\]

polynomial of degree \( n \).

The function \( T_n(x) = \cos((n+1)\cos x) \) is known as the Chebyshev polynomial of degree \( n \). These functions play a very important role in numerical analysis.