Function approximation

Polynomial interpolation mainly has applications in function approximation, with respect to some norm:

For functions, some example norms are:

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$

$$\|f\|_2 = \sqrt{\int_a^b |f(x)|^2\,dx}$$

$$\|f\|_1 = \int_a^b |f(x)|\,dx$$

Just like for n-dimensional vectors.

Norms of functions satisfy the same properties as those in the finite dimensional vector case:

1. $\|f\| \geq 0$, $\|f\|=0$ iff $f=0$
2. $\|cf\| = |c|\,\|f\|$
3. $\|f+g\| \leq \|f\| + \|g\|$

Ex: The 2norm of a function can be generalized by introducing a "weight" function $w>0$:

$$\|f\|_{2,w} = \sqrt{\int_a^b |f(x)|^2\, w(x)\,dx}$$

So: the polynomial p_n of degree n that best approximates a function f in the ∞-norm is

$$\underbrace{\min_{p_n\in P_n} \|p_n - f\|_\infty}_{\text{maximum pointwise error.}}$$

Do not think of p_n as a polynomial interpolant of f.

From analysis class, we know that continuous functions f on some finite interval can be approximated <u>arbitrarily</u> well by a polynomial of "some" degree: this result is known as the <u>Weierstrass Approximation Theorem</u>.

I.e. For any $\epsilon > 0$, there exists a polynomial p such that $\| f - p \|_\infty < \epsilon$, ~~[illegible]~~

Unfortunately, this is a completely useless theorem for numerical approximation.

It doesn't tell you how to find p!

The question of restricting $p \in P_n$ is much more interesting, and actually useful.

To pose the problem:

For $n > 0$, find $p_n \in P_n$ such that

$$\| f - p_n \|_\infty = \min_{q \in P_n} \| f - q \|_\infty .$$

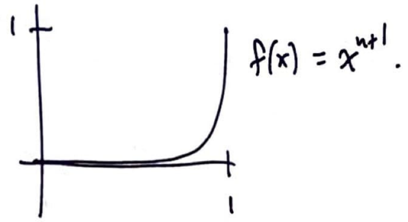
<u>Theorem</u>: Such a p_n exists, and is <u>unique</u>.

(The proof does <u>not</u> tell us how to find p_n.)

In general, one cannot write down the <u>minimax</u> polynomial, i.e. the polynomial p_n such that

$$\| f - p_n \|_\infty = \min_{q \in P_n} \max_{x \in [a,b]} | f(x) - q(x) |$$

However, we can explicitly write down then minimax polynomial approximation to the monomial $f(x) = x^{n+1}$ on $[0,1]$



Theorem Let $n \geq 0$, then $\|p_n - f\|_\infty$, with $f(x) = x^{n+1}$, is minimized when $p_n(x) = \underbrace{x^{n+1} - \frac{1}{2^n}\cos\left((n+1)\operatorname{acos} x\right)}_{\text{polynomial of degree } n}.$

The function $T_n(x) = \cos\left(n \operatorname{acos} x\right)$ is known as the Chebyshev polynomial of degree n. These functions play a very important role in numerical analysis.