Chebyshev polynomials

$T_n(x) = \cos(n \operatorname{acos} x)$ $\quad n = 0, 1, 2, \ldots$

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = \cos(2 \operatorname{acos} x)$

$\vdots$

$\underbrace{T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x)}_{\text{can prove using trig identities}}$ $\leftarrow$ polynomial of degree $n+1$

applied to this definition

Usually only concerned with $T_n(x)$ for $x \in [-1,1]$.

Trivially, the zeros of T_n can be computed as:

$$\cos(\underbrace{n \operatorname{acos} x}) = 0$$

$$\Rightarrow \quad n \operatorname{acos} x = \frac{\pi}{2}(2m+1) \qquad \text{for } m = \dots -2, -1, 0, 1, 2 \dots$$

$$\operatorname{acos} x = \frac{\pi}{2n}(2m+1)$$

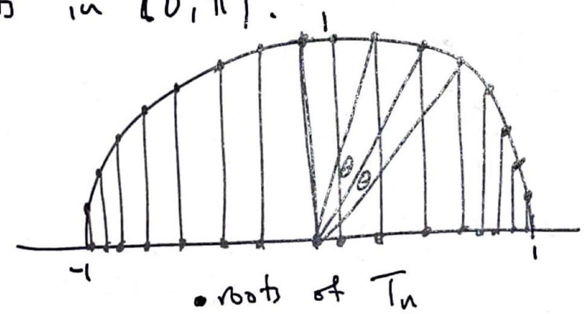
$$x = \cos\left(\frac{(2m+1)(\pi)}{2n}\right) \qquad m = 0, 1, \dots, n-1 \quad \left(\text{roots repeat for } m \geq n\right).$$

The roots on $[-1,1]$ can be ordered from $[-1,1]$ as:

$$x_j = -\cos\left(\underbrace{\frac{(2j-1)\pi}{2n}}\right) \qquad j = 1, \dots, n \quad (n \text{ roots}).$$

this is an angle, as $j = 1 \dots n$, we get equispaced points in $(0, \pi)$.

$\Rightarrow$



<u>Claim</u>: Interpolation of a function f on $[a,b]$ with a degree n polynomial p_n at the roots of T_n – i.e. Chebyshev nodes – yields a <u>near</u> <u>minimax</u> polynomial approximant.

<u>Idea</u>: The approximation error of the interpolation is

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j)$$

If x_j are chosen to be the roots of T_n (properly scaled to this interval), then $\underbrace{\prod_{j=0}^{n}(x-x_j)}_{} = \frac{1}{2^n} T_n(x)$

Factorization of $\frac{1}{2^n} T_n(x)$.

What is special about $\frac{1}{2^n} T_n(x)$?

It can be shown that $\frac{1}{2^n} T_n$ is the minimum norm monic polynomial. A monic polynomial of degree n is one whose coefficient on the x^n term is 1.

$$\|f - p_n\|_\infty \leq \frac{M_{n+1}}{(n+1)!} \underbrace{\left\| \prod_{j=0}^{n} (x-x_j) \right\|_\infty}_{\text{just minimize this by choosing the "best" } x_j\text{'s.}}$$

Approximation in the 2-norm

The 2-norm of a function, with some general continuous weight function $w>0$, on (a,b) is:

$$\|f\|_2^2 = \int_a^b (f(x))^2 \, w(x) \, dx.$$

Goal: Find $p_n \in P_n$ such that $\|f-p_n\|_2$ is minimized. This is a least squares approximation to f – exactly analogous to solving least squares problems in finite dimensions (ie. Linear Alg.)

Therefore the solution is obtained by computing the orthogonal projection of f onto the space P_n, under the inner product:

$$(f,g)_w = \text{inner product of } f \text{ with } g$$

$$= \int_a^b f(x) \, g(x) \, w(x) \, dx.$$

Easy to check that this is indeed an inner product

There is no reason why the best p_n for the 2-norm error is the same p_n for the ∞-norm error. In more detail...