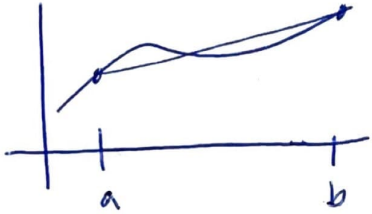


Honors Numerical Analysis

Lecture 20

Error in the Trapezoidal Rule:-



The interpolant p_1 is given by

$$p_1(x) = f(a) \frac{x-b}{a-b} + f(b) \frac{x-a}{b-a}$$

The error for this interpolant is given from an analysis of Taylor's Thm (from earlier):

$$f(x) - p_1(x) = \frac{1}{2!} f''(\xi) (x-a)(x-b)$$

↑
some value in $[a, b]$

$$\text{Let } \eta = \operatorname{argmax}_x |f''(x)|$$

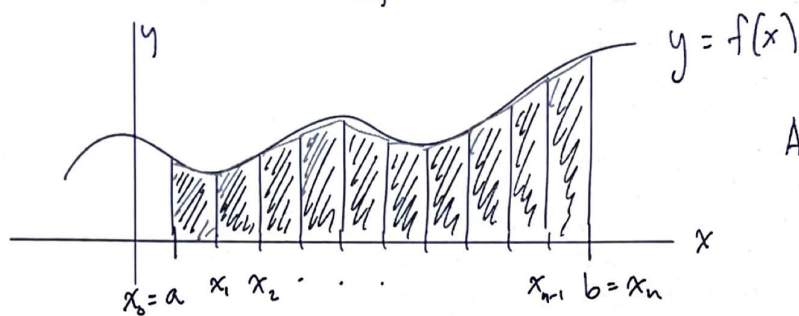
$$\text{Then } \left| \int_a^b f(x) - \int_a^b p_1(x) \right| = \left| \int_a^b \frac{1}{2!} f''(\xi) (x-a)(x-b) dx \right|$$

$$\leq \frac{1}{2} f''(\eta) \left| \int_a^b (x-a)(x-b) dx \right|$$

$$= \frac{1}{12} f''(\eta) (b-a)^3.$$

call this h^3 , $h = b-a$

The Composite Trapezoidal Rule



Apply the trapezoidal rule on each smaller interval (analogous to the Riemannian Integral).

$$x_j = a + jh \\ = a + j\left(\frac{b-a}{n}\right)$$

Then, the integral is approximated by:

$$\int_a^b f(x) dx \approx \underbrace{T_n f}_{N\text{-interval Trapezoidal Rule}} \\ = \sum_{j=1}^n h \left(\frac{f(x_{j-1}) + f(x_j)}{2} \right) \\ = h \left(\sum_{j=0}^n f(x_j) - \frac{1}{2}(f(a) + f(b)) \right)$$

Question: What is the error $\left| \int_a^b f(x) dx - T_n f \right|$?

This error can be shown to be $O(h^2)$.

Idea of the proof: on a single interval $[a, b]$, Taylor expand f about $x = \frac{a+b}{2}$ (the midpoint). Apply the Trapezoidal rule to this Taylor series.

In fact, a much more powerful formula exists that characterizes the error explicitly.

The Euler-Maclaurin Expansion

Theorem: Let $f \in C^{2k}[a,b]$, and $[a,b]$ be divided into n equal subintervals, $[x_{j-1}, x_j]$, with $x_j = a + jh$.

Then, $\int_a^b f(x) dx - T_n f$

$$I - T_n f = \sum_{r=1}^k c_r h^{2r} \left(f^{(2r-1)}(b) - f^{(2r-1)}(a) \right) - d_{2k} \left(\frac{h}{2} \right)^{2k}.$$
$$= \frac{h^2}{12} (f'(b) - f'(a)) - \frac{h^4}{720} (f^{(3)}(b) - f^{(3)}(a)) + \dots + (-1)^{k-1} \frac{B_{2k}}{(2k)!} h^{2k} f^{(2k)}(\xi)$$

$\xi \in [a,b]$

These coefficients are given by:

$$c_r = -\frac{B_{2r}}{(2r)!}, \quad B_{2r} \text{ is a Bernoulli number:}$$

$$\frac{x}{2} \coth\left(\frac{x}{2}\right) = \sum_{r=0}^{\infty} \frac{B_{2r}}{(2r)!} x^{2r}.$$

The calculation of B_{2r} was the output of arguably the first "computer program" written by Ada Lovelace and Charles Babbage.

Implications of Euler-Maclaurin

If $f \in C^\infty[a,b]$ and periodic with $f^{(j)}(a) = f^{(j)}(b)$ (think Fourier Series, or $\cos mx$, $\sin mx$, etc.) then this error $|I - T_n f|$ decays superalgebraically as $n \rightarrow \infty$.

Def: $\epsilon_n \rightarrow 0$ superalgebraically if

$$\lim_{n \rightarrow \infty} \frac{\epsilon_n}{h^p} = 0 \quad \text{for any } p > 0.$$

This means that $\epsilon_n \rightarrow 0$ faster than any power of h .

For this reason, the trapezoidal rule is very important in various numerical methods.

Ex: $J_0(x) = \frac{1}{\pi} \int_0^\pi e^{ix \cos \theta} d\theta.$

$e^{ix \cos \theta}$ is not periodic on $[0, \pi]$, but it is on $[0, 2\pi]$.

It can be shown that

$$J_0(x) = \frac{1}{\pi} \int_0^\pi e^{ix \cos \theta} d\theta = \underbrace{\frac{1}{2\pi} \int_0^{2\pi} e^{ix \cos \theta} d\theta}_{\text{See mybessel.m for computing this via the trapezoidal rule.}}$$

Clebshaw - Curtis Quadrature

Special case of Newton-Cotes Formula:

- Interpolate f at Chebyshev nodes
- Integrate each Chebyshev polynomial:

Details Approximate $\int_{-1}^1 f(x) dx$ by

$$\approx \int_{-1}^1 p_n(x) dx \quad \text{where } p_n \text{ is the } \mathcal{L}_n \text{ interpolant expressed in Chebyshev polynomials:}$$

$$f(x) \approx p_n(x) \\ = \sum_{j=0}^n c_j T_j(x)$$

and then $\int_{-1}^1 f(x) dx \approx \sum_{j=0}^n c_j \int_{-1}^1 T_j(x) dx$

$$= \sum_{j=0}^n c_j \int_{-1}^1 \cos(j \cdot \arccos x) dx$$

$$\text{Let } \arccos x = \theta \\ dx = -\sin \theta$$

$$= \sum_{j=0}^n c_j \int_0^\pi \cos(j \cdot \theta) \sin \theta d\theta$$

$$= \sum_{j=0}^n c_j \int_0^\pi \cos(j \cdot \theta) \sin \theta d\theta$$

$$= \sum_{j=0}^n c_j \frac{1 + \cos(j\pi)}{1 - j^2}$$

$$= \sum_{j=0}^{n/2} c_{2j} \frac{2}{1 - (2j)^2}$$

Equivalently, we can think about the analogous calculation:

$$\text{Let } x = \cos \theta,$$

$$\text{then } f(x) = \sum c_k T_k(x)$$

$$\Rightarrow f(\cos \theta) = \sum c_k \underbrace{\cos(k \arccos x)}_{=\theta}$$

$$= \sum c_k \cos(k\theta)$$

← this is a "cosine" series for the function $f(\cos \theta)$.

$$\text{And then: } \int_{-1}^1 f(x) dx \approx \int_0^\pi f(\cos \theta) \sin \theta d\theta$$

$$\approx \int_0^\pi (\sum c_k \cos k\theta) \sin \theta d\theta$$

= same answer as before.

We will return to a "fast" method for computing the c_k 's when we talk about the Discrete Fourier Transform, but for now, you can imagine computing them with the composite trapezoidal rule since $f(\cos \theta) \sin \theta$ is a periodic function.