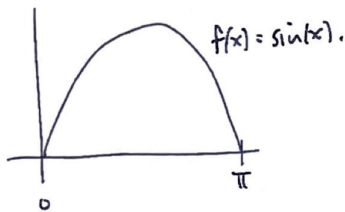# Approximation in L^2 using orthogonal polynomials

Example Compute the best quadratic 2-norm approximation
to $f(x) = \sin x$ on $(0, \pi)$ with weight function $w(x) = 1$.

$f(x) = \sin(x)$.

The first 3 Legendre polynomials on
$(-1, 1)$ are:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = x^2 - \frac{1}{3}$$

To define these functions on $(0, \pi)$, let $t = \frac{x+1}{2}\pi$

$$\Rightarrow \quad x = \frac{2t}{\pi} - 1$$

So the shifted orthogonal polynomials are:

$$\tilde{P}_0(t) = 1$$
$$\tilde{P}_1(t) = x = \frac{2}{\pi}t - 1 = \frac{2}{\pi}\left(t - \frac{\pi}{2}\right)$$
$$\tilde{P}_2(t) = x^2 - \frac{1}{3} = \left(\frac{2t}{\pi} - 1\right)^2 - \frac{1}{3}$$

$$= \frac{4t^2}{\pi^2} - \frac{4t}{\pi} + 1 - \frac{1}{3}$$

$$= \frac{4}{\pi}\left(\frac{t^2}{\pi} - t + \frac{\pi}{6}\right).$$

The best approximation is given by the projection of f
onto $\text{span}\{\tilde{P}_0, \tilde{P}_1, \tilde{P}_2\}$. Let $q =$ best 2nd-degree pol. approximat.

$$\Rightarrow q = \frac{(f, \tilde{P}_0)}{(\tilde{P}_0, \tilde{P}_0)}\tilde{P}_0 + \frac{(f, \tilde{P}_1)}{(\tilde{P}_1, \tilde{P}_1)}\tilde{P}_1 + \frac{(f, \tilde{P}_2)}{(\tilde{P}_2, \tilde{P}_2)}\tilde{P}_2.$$

So $(f, \tilde{P}_b)$ can be computed using:

$$\int_0^\pi x \sin x \, dx = \sin x - x \cos x.$$
$$\int_0^\pi x^2 \sin x \, dx = 2x \sin x - (x^2 - 2)\cos x.$$

A bit about orthogonal polynomials

All orthogonal polynomials are eigenfunctions of a differential operator:

$$\mathcal{L}u = -\frac{1}{w}(p u')' + qu \quad , \quad \text{with } p, q, w \text{ functions}$$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}$

Sturm-Liouville Operator

It can be shown that $\mathcal{L}$ is a self-adjoint $\overset{\text{show by}}{\underset{}{\longleftarrow}}$ $(\mathcal{L}u, v) = (u, \mathcal{L}v)$ + boundary conditions

linear operator (given suitable boundary conditions), and therefore it has real eigenvalues:

$$\mathcal{L}u = \lambda u \quad , \quad \lambda \text{ is real.}$$

$\Rightarrow$ The eigenfunctions are orthogonal with respect to the inner product $(u,v) = \int u(x) v(x) w(x) \, dx$.

Ex: Legendre polynomials

P_n satisfies the Sturm-Liouville eigenfunction problem:

$$\frac{d}{dx}\left((1-x^2)\frac{d}{dx} \right) P_n(x) = -n(n+1) P_n(x).$$

Numerical Evaluation of Orth. Pols

All orthogonal polynomials ~~also~~ satisfy a "three-term-recurrence" formula of the form:

$$q_n(x) = (x + a_n) q_{n-1}(x) + b_n q_{n-2} \qquad \text{for } n = 2, 3, \dots$$

$$q_0(x) = 1$$

$$q_1(x) = x + a_1$$

> Alternative form:
> $$q_{n+1} = a_n x \, q_n + b_n q_{n-1}$$

Since they are orthogonal, the coefficients are given by

$$a_n = - \frac{(x P_{n-1}, P_{n-1})}{(P_{n-1}, P_{n-1})}$$

$$b_n = - \frac{(P_{n-1}, P_{n-1})}{(P_{n-2}, P_{n-2})}$$

EX: Legendre:
$$P_{n+1}(x) = \frac{2n+1}{n+1} x \, P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

Chebyshev:
$$T_{n+1}(x) = 2x \, T_n(x) - T_{n-1}(x)$$

Hermite:
$$H_{n+1}(x) = 2x \, H_n(x) - 2n \, H_{n-1}(x)$$

Stability comes from an analysis of
$$\begin{pmatrix} 0 & 1 \\ b_n & a_n x \end{pmatrix} \begin{pmatrix} q_{n-1} \\ q_n \end{pmatrix} = \begin{pmatrix} q_n \\ q_{n+1} \end{pmatrix}$$

Numerical Integration

Almost no integral can be computed analytically, they must be evaluated numerically.

Ex:

$$\underbrace{erf(x)}_{error\ function} = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Application ODES, initial value problems

$$(*) \quad y'(t) = f(t)$$
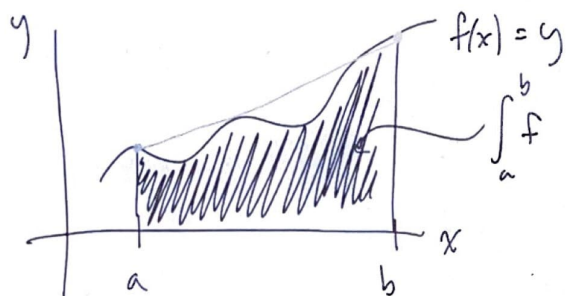$$y(0) = y_0$$

The analytic solution is

$$y(t) = y_0 + \int_0^t f(\tau) d\tau$$

All numerical methods for solving $(*)$ are based on numerically approximating the integral.

More to come later...

The Trapezoidal (Trapezium) Rule

The most basic numerical integration technique:

Option 1: Approximate f on $[a,b]$ by a line, integrate the line.

Area under this line is

$$\underbrace{(b-a)}_{\text{width}} \underbrace{\frac{\left(f(a) + f(b)\right)}{2}}_{\substack{\text{average of edge} \\ \text{heights.}}} \approx \int_a^b f(x)\, dx.$$

Area of a trapezoid.

① f was approximated by a line which interpolated f at $x = a$ and $x > b$.

② This interpolating line was integrated.

The trapezoidal rule is a special case of more general numerical integration formulas known as Newton-Cotes rules:

Idea Interpolate f on $[a,b]$, and then integrate the interpolating polynomial.

This can be done with arbitrarily high degree (high order interpolation) but remember: the interpolating polynomial may suffer from Runge's Phenomena.

In general: Interpolate f at the points $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$

$$p_n(x) = \sum_{j=0}^{n} L_j(x) \, f(x_j)$$

$\qquad$ Lagrange function $L_j(x) = \prod_{i \neq j} \dfrac{x - x_i}{x_j - x_i}$

Then the integral of f can be approximated as:

$$\int_a^b f(x) \, dx \approx \int_a^b p_n(x) \, dx$$

$$= \int_a^b \sum_{j=0}^{n} L_j(x) \, f(x_j) \, dx$$

$$= \sum_{j=0}^{n} \left(\underbrace{\int_a^b L_j(x) \, dx}_{w_j} \right) f(x_j)$$

$$= \sum_{j=0}^{n} w_j \, f(x_j) \quad \Big\} \text{ Standard form of a } \underline{\text{Quadrature Rule}} \text{ (Numerical)}$$

$w_j =$ quadrature weights

$x_j =$ quadrature nodes (the locations where f is evaluated).

<u>Note</u>: $w_j = \displaystyle\int_a^b L_j(x) \, dx$ only depends on the location of the quadrature nodes, <u>not</u> the values of f.

<u>Note 2</u>: Warning: Large n may result in an inaccurate approximation by the interpolating polynomial due to Runge's Phenomena.

$\qquad$ <u>Remedy</u>: Use many smaller lower-order interpolations, glue the result together.

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