<u>Our goal</u>: For a function $f \in L^2_w(a,b)$ $\left(\|f\|_2 = \sqrt{\int_a^b (f(x))^2 w(x)\, dx} < \infty \right)$

find $p_n \in \mathbb{P}_n$ such that:

$$\|f - p_n\|_2^2 = \inf_{q \in \mathbb{P}_n} \|f - q\|_2^2$$

$$= \inf_{q \in \mathbb{P}_n} \int_a^b (f(x) - q(x))^2 w(x)\, dx.$$

<u>Ex</u>: Let $n = 0$, $f(x) = -2x^2$ on $[-1, 1]$, $w(x) = 1$.

Find $p_0 = c$ to minimize $\|f - p_0\|_2^2$.

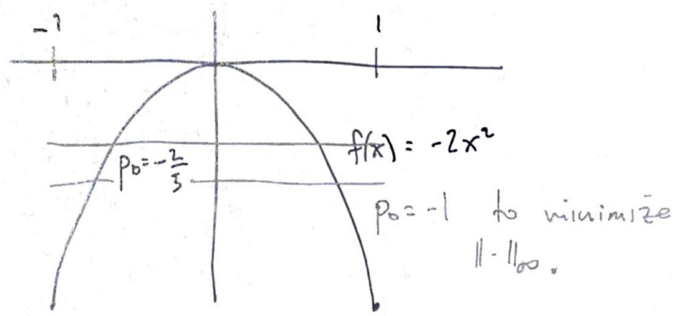
$$\|f - p_0\|_2^2 = \int_{-1}^{1} (-2x^2 - c)^2\, dx$$

$$= 2c^2 + \frac{8c}{3} + \frac{8}{5}$$

Find c to minimize this quantity.

$$\frac{d}{dc} \|f - p_0\|_2^2 = 4c + \frac{8}{5} = 0$$

$$\Rightarrow c = -\frac{2}{3}$$



What would the best approximation in $\|\cdot\|_\infty$ be?

<u>Computing the best 2-norm approximation</u>

<u>Recall</u>: The least squares solution to $A\vec{x} = \vec{b}$ is obtained by solving $A\vec{x} = \underbrace{QQ^T\vec{b}}$

projection of $\vec{b}$ onto col A, Q is obtained by applying the Gram-Schmidt process to the columns of A.

To apply Gram-Schmidt, all that was needed was a vector space and an inner product.

We can do the same thing for polynomial least squares:

1. Trivially, P_n is a vector space.
2. We can define an inner product on P_n by: $(f,g) = \int_a^b f(x)\, g(x)\, w(x)\, dx$.

Two functions are orthogonal if $(f,g) = 0$.

Let $p_0, p_1, p_2, \dots, p_n$ be a basis for P_n.

The 2-norm approximation problem takes the form:

For $q(x) = \sum_{j=0}^{n} c_j p_j(x)$, $\|f-q\|_2^2$ is given by: (set $w=1$ for now)

$$A = \int_a^b \left(f(x) - \sum_{j=0}^{n} c_j p_j(x)\right)^2 dx$$

$$= \int_a^b (f(x))^2\, dx - 2\sum_{j=0}^{n} c_j \int_a^b f(x)\, p_j(x)\, dx + \sum_{j=0}^{n}\sum_{k=0}^{n} c_j c_k \int_a^b p_j(x)\, p_k(x)\, dx$$

$$= (f,f) - 2\sum_{j=0}^{n} c_j (f, p_j) + \sum_{j=0}^{n}\sum_{k=0}^{n} c_j c_k (p_j, p_k)$$

Writing down

$\nabla A = \vec{0}$, we have that:

$$\frac{\partial A}{\partial c_0} = 0, \quad \frac{\partial A}{\partial c_1} = 0, \quad \dots, \quad \frac{\partial A}{\partial c_n} = 0$$

$$\Rightarrow \frac{\partial A}{\partial c_\ell} = -2(f, p_\ell) + 2\sum_{k=0}^{n} c_k (p_\ell, p_k) = 0$$

$$\Rightarrow \sum_{k=0}^{n} c_k (p_\ell, p_k) = (f, p_\ell)$$ $\leftarrow$ $n+1$ linear equations, in $n+1$ variables $c_0 \dots c_n$.

In matrix form:

$$\begin{pmatrix} (p_0,p_0) & \cdots & (p_0,p_n) \\ (p_1,p_0) & \cdots & (p_1,p_n) \\ \vdots & & \\ (p_n,p_0) & \cdots & (p_n,p_n) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} (f,p_0) \\ (f,p_1) \\ \vdots \\ (f,p_n) \end{pmatrix}$$

Therefore once this linear system has been solved, the best 2-norm approximation to f is:

$$q = c_0 p_0 + c_1 p_1 + \dots + c_n p_n .$$

If the p_ℓ's were orthonormal, i.e. $(p_\ell, p_k) = \delta_{\ell k}$, then the above system is of the form:

$$I \vec{c} = RHS.$$

So $=$ $c_\ell = (f, p_\ell)$. (if p_ℓ's are orthonormal).

Function approximation in the 2-norm.

<u>Goal</u> For a given function f on $[a,b]$, find $p_n \in P_n$ that $\|f - p_n\|_2$ is as small as possible.

$\Rightarrow$ We previously derived that this is a least squares problem and therefore there exists a constructive solution, i.e.

If $p_n(x) = c_0\varphi_0(x) + c_1\varphi_1(x) + \dots + c_n\varphi_n(x)$

form a basis for P_n, then the coefficients c_j are obtained by solving a linear system.

Finally, <u>if</u> the functions φ_j are orthonormal, then the coefficients c_j are merely inner products:

$$c_j = (f, \varphi_j). \quad \text{(analogous to Gram-Schmidt, or orthogonal projections)}.$$

This approximation of f is equivalent to finding its orthogonal projection onto P_n under the inner product:

$$(f,g) = \int_a^b f(x)\, g(x)\, w(x)\, dx.$$

<u>Definition</u> If the sequence of polynomials $\varphi_0, \varphi_1, \dots, \varphi_n$, with $\deg \varphi_j = j$, on the interval (a,b) satisfies

$$\int_a^b \varphi_j(x)\, \varphi_k(x) = 0 \quad \text{if } j \neq k,$$

then $\varphi_0, \dots, \varphi_n$ is a <u>system of orthogonal polynomials</u> (with weight $w(x) = 1$).

(Likewise we would define $(f,g) = \int_a^b f(x)\, g(x)\, w(x)\, dx$.)

Example: Find q_0, q_1, q_2 on $[-1,1]$ with weight function $w(x) = 1$.

Set $q_0(x) = 1$.

$q_1(x) = ax + b$.

If $(q_0, q_1) = 0$ then $\int_{-1}^{1} 1 \cdot (ax+b)\, dx = 0$

$2b = 0 \Rightarrow b = 0$.

Set $q_1(x) = x$.

Let $q_2(x) = x^2 + bx + c$

Two conditions must be satisfied:

$\int_{-1}^{1} q_0(x) q_2(x)\, dx = 0$

$\int_{-1}^{1} (x^2 + bx + c)\, dx = 0$

$\frac{2}{3} + 2c = 0$

$c = \frac{-1}{3}$

$\int_{-1}^{1} q_1(x) q_2(x)\, dx = 0$

$\int_{-1}^{1} x(x^2 + bx + c)\, dx = 0$

$\frac{2}{3} b = 0 \Rightarrow b = 0$.

So we set $q_2(x) = x^2 - \frac{1}{3}$.

So by construction, 1, x, and $x^2 - \frac{1}{3}$ are orthogonal on $[-1,1]$.

We could do the same calculations for $q_3, q_4, \ldots$

The resulting polynomials are known as Legendre Polynomials. They form an orthogonal basis for all of $\underbrace{L^2[-1,1]}$ under the inner product $(f,g) = \int_{-1}^{1} f(x) g(x)\, dx$.

$f \in L^2[-1,1]$ iff $\underbrace{\int_{-1}^{1} (f(x))^2\, dx < \infty}_{\|f\|_2^2 < \infty}$.

Legendre polynomials can also be constructed another way as well: the Gram-Schmidt process.

Start with $P_0(x) = 1$, $P_1(x) = x$. $\Big]$ automatically orthogonal.

Set $m_2(x) = x^2$. $\leftarrow$ linearly independent from P_0, P_1.

For Gram-Schmidt: $P_2 = m_2 - \mathrm{Proj}_{\{P_0, P_1\}} m_2$

$$= m_2 - \frac{(m_2, P_0)}{(P_0, P_0)} P_0 - \frac{(m_2, P_1)\, P_1}{(P_1, P_1)}$$

So $P_2(x) = x^2 - \frac{2}{3}\frac{1}{2}\cdot 1 - 0$

$= x^2 - \frac{1}{3}$ Exactly the same as before.

So in general, compute

$$P_n = m_n - \sum_{l=0}^{n-1} \frac{(m_n, P_l)}{(P_l, P_l)} P_l$$

$$P_n(x) = x^n - \sum_{l=0}^{n-1} \frac{P_l(x)}{\|P_l\|_2^2} \int_{-1}^{1} x^n P_l(x)\, dx$$

These polynomials can be scaled to any interval.

If $\int_{-1}^{1} P_j(x) P_k(x)\, dx = 0$ if $j \neq k$, then it is easy to show that $\int_a^b P_j(t) P_k(t)\, dt = 0$ if $j \neq k$

where $t = \left(\frac{x+1}{2}\right)(b-a) + a$.

Ex: Chebyshev Polynomials

We know that $\int_0^{\pi} \cos mt \cos nt\, dt = 0$ if $m \neq n$.

Let $t = a\cos x$, $dt = \frac{1}{\sqrt{1-x^2}} dx$, $t: -1 \to 1$

$$\Rightarrow \int_{-1}^{1} \underbrace{\cos(m \operatorname{acos} x)}_{T_m(x)} \cdot \underbrace{\cos(n \operatorname{acos} x)}_{T_n(x)} \cdot \frac{dx}{\sqrt{1-x^2}} = 0 \quad \text{if } m \neq n.$$

$$\Rightarrow \int_{-1}^{1} T_m(x)\, T_n(x) \frac{1}{\sqrt{1-x^2}}\, dx = 0 \quad \text{if } m \neq n.$$

Therefore, the functions $T_0, T_1, T_2, \ldots$ are orthogonal on $[-1,1]$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$.

<u>Theorem</u> If $\int_a^b |f(x)|^2 w(x)\, dx < \infty$ (ie. $f \in L^2_w[a,b]$), there is a unique degree n polynomial p_n such that

$$\| f - p_n \|_{w,2} = \min_{q \in P_n} \| f - q \|_{w,2}$$

where $\| f \|^2_{w,2} = \int_a^b |f(x)|^2 w(x)\, dx$.

<u>Proof</u>: Gram-Schmidt, solve directly for the coefficients of the associated orthogonal polynomial expansion to compute the approximation.

Note: This is just linear algebra!

Famous sets of orthogonal polynomials:

$w(x)$	(a,b)	Polynomial
1	$(-1,1)$	Legendre
$1/\sqrt{1-x^2}$	$(-1,1)$	Chebyshev
e^{-x}	$(0,\infty)$	Laguerre
e^{-x^2}	$(-\infty,\infty)$	Hermite.