Function approximation

Honors Numerical Analysis

Lecture 17

Polynomial interpolation mainly has applications in function approximation, with respect to some norm:

For functions, some example norms are:

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$

$$\|f\|_2 = \sqrt{\int_a^b |f(x)|^2\,dx}$$

$$\|f\|_1 = \int_a^b |f(x)|\,dx$$

Just like for n-dimensional vectors.

Norms of functions satisfy the same properties as those in the finite dimensional vector case:

① $\|f\| \geq 0$, $\|f\| = 0$ iff $f = 0$

② $\|cf\| = |c|\,\|f\|$

③ $\|f+g\| \leq \|f\| + \|g\|$

Ex: The 2norm of a function can be generalized by introducing a "weight" function $w > 0$:

$$\|f\|_{2,w} = \sqrt{\int_a^b |f(x)|^2\, w(x)\,dx}$$

So: the polynomial p_n of degree n that best approximates a function f in the ∞-norm is

$$\min_{p_n \in P_n} \underbrace{\|p_n - f\|_\infty}_{\text{maximum pointwise error.}}$$

Do not think of p_n as a polynomial interpolant of f.

From analysis class, we know that continuous functions f on some finite interval can be approximated arbitrarily well by a polynomial of "some" degree: this result is known as the Weierstrass Approximation Theorem.

I.e. For any $\epsilon > 0$, there exists a polynomial p such that $\| f - p \|_\infty < \epsilon$,

Unfortunately, this is a completely useless theorem for numerical approximation.
It doesn't tell you how to find p!

The question of restricting $p \in P_n$ is much more interesting, and actually useful.

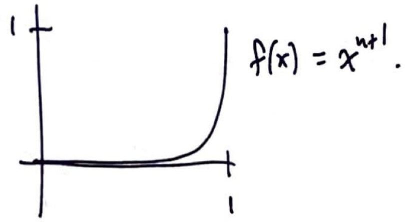
To pose the problem:

For $n > 0$, find $p_n \in P_n$ such that
$$\| f - p_n \|_\infty = \min_{q \in P_n} \| f - q \|_\infty .$$

Theorem: Such a p_n exists, and is unique.
(The proof does not tell us how to find p_n.)

In general, one cannot write down the minimax polynomial, i.e. the polynomial p_n such that
$$\| f - p_n \|_\infty = \min_{q \in P_n} \max_{x \in [a,b]} | f(x) - q(x) |$$

However, we can explicitly write down then minimax polynomial approximation to the monomial $f(x) = x^{n+1}$ on $[0,1]$



Theorem Let $n \geq 0$, then $\|p_n - f\|_\infty$, with $f(x) = x^{n+1}$, is minimized when $p_n(x) = \underbrace{x^{n+1} - \frac{1}{2^n}\cos\left((n+1)\,\mathrm{acos}\,x\right)}_{\text{polynomial of degree } n.}$.

The function $T_n(x) = \cos(n\,\mathrm{acos}\,x)$ is known as the Chebyshev polynomial of degree n. These functions play a very important role in numerical analysis.

Chebyshev polynomials

$T_n(x) = \cos(n \operatorname{acos} x)$ $\qquad n = 0, 1, 2, \ldots$

$T_0(x) = 1$

$T_1(x) = x$

Usually only concerned with $T_n(x)$ for $x \in [-1, 1]$.

$T_2(x) = \cos(2 \operatorname{acos} x)$

$\vdots$

$\underbrace{T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x)}_{\text{can prove using trig identities}}$ ← polynomial of degree $n+1$

applied to this definition

Trivially, the zeros of T_n can be computed as:

$$\cos(\underbrace{n \operatorname{acos} x}) = 0$$

$$\Rightarrow \quad n \operatorname{acos} x = \frac{\pi}{2}(2m+1) \quad \text{for } m = \ldots -2, -1, 0, 1, 2 \ldots$$

$$\operatorname{acos} x = \frac{\pi}{2n}(2m+1)$$

$$x = \cos\left(\frac{(2m+1)(\pi)}{2n}\right) \quad m = 0, 1, \ldots, n-1 \quad (\text{roots repeat for } m \geq n).$$

The roots on $[-1,1]$ can be ordered from $[-1,1]$ as:

$$x_j = -\cos\left(\underbrace{\frac{(2j-1)\pi}{2n}}\right) \quad j = 1, \ldots, n \quad (n \text{ roots}).$$

this is an angle, as $j = 1 \ldots n$, we get equispaced points in $(0, \pi)$.

$\Rightarrow$

• roots of T_n

<u>Claim</u>: Interpolation of a function f on $[a,b]$ with a degree n polynomial p_n at the roots of T_n – i.e. Chebyshev nodes – yields a <u>near</u> <u>minimax</u> polynomial approximant.

<u>Idea</u>: The approximation error of the interpolation is

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j)$$

If x_j are chosen to be the roots of T_n (properly scaled to this interval), then $\underbrace{\prod_{j=0}^{n} (x - x_j)} = \frac{1}{2^n} T_n(x)$

Factorization of $\frac{1}{2^n} T_n(x)$.

What is special about $\frac{1}{2^n} T_n(x)$?

It can be shown that $\frac{1}{2^n} T_n$ is the minimum norm monic polynomial. A monic polynomial of degree n is one whose coefficient on the x^n term is 1.

$$\|f - p_n\|_\infty \leq \frac{M_{n+1}}{(n+1)!} \underbrace{\left\| \prod_{j=0}^{n} (x - x_j) \right\|_\infty}_{\text{just minimize this by choosing the "best" } x_j\text{'s.}}$$

Approximation in the 2-norm

The 2-norm of a function, with some general continuous weight function $w>0$, on (a,b) is:

$$\|f\|_2^2 = \int_a^b (f(x))^2 \, w(x) \, dx.$$

Goal: Find $p_n \in P_n$ such that $\|f - p_n\|_2$ is minimized. This is a least squares approximation to f – exactly analogous to solving least squares problems in finite dimensions (ie. Linear Alg.)

Therefore the solution is obtained by computing the orthogonal projection of f onto the space P_n, under the inner product:

$$(f,g)_w = \text{inner product of } f \text{ with } g$$

$$= \int_a^b f(x) \, g(x) \, w(x) \, dx.$$

Easy to check that this is indeed an inner product

There is no reason why the best p_n for the 2-norm error is the same p_n for the ∞-norm error. In more detail ...