Honors Numerical Analysis

Lecture 5

Vector and Matrix Norms

Def: Let $\vec{u}, \vec{v}, \vec{x} \in \mathbb{C}^n$. $\|\cdot\|$ is a norm on $\mathbb{C}^n$ if

(1) $\|\vec{x}\| \geq 0$, $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$.

(2) $\|\alpha \vec{x}\| = |\alpha| \, \|\vec{x}\|$ for $\alpha \in \mathbb{C}$

(3) $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ (triangle inequality)

Think about norms as distances, for example:

$$\|\vec{u}\|_2 = \sqrt{\vec{u} \cdot \vec{u}} = \sum_{i=1}^{n} |u_i|^2 \quad \leftarrow l_2 \text{ norm.}$$

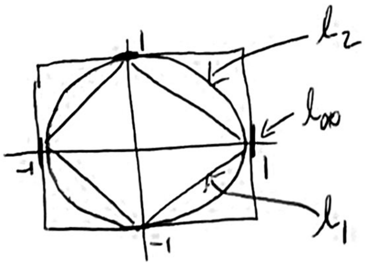
Other norms commonly used

l_∞ norm: $\|\vec{u}\|_\infty = \max_i |u_i|$

l_1 norm: $\|\vec{u}\|_1 = \sum_i |u_i|$

l_p norm: $\|\vec{u}\|_p = \left(\sum_i |u_i|^p \right)^{1/p}$

Ex: Unit circles in $1, \infty$, and 2-norms:



Only the l_2 norm comes from an inner product

Similarly, matrix norms must satisfy similar conditions:

Def For $A, B \in \mathbb{C}^{m \times n}$, $\|\cdot\|$ is a matrix norm if

(1) $\|A\| \geq 0$, $\|A\| = 0$ iff $A = 0$

(2) $\|\alpha A\| = |\alpha| \, \|A\|$ for $\alpha \in \mathbb{C}$

(3) $\|A+B\| \leq \|A\| + \|B\|$

[(4) $\|AB\| \leq \|A\| \cdot \|B\|$, sometimes useful]

If $\|\cdot\|$ is <u>any</u> vector norm, then the induced matrix norm is

$$\|A\| = \max_{\|\vec{u}\|=1} \|A\vec{u}\| = \max_{\vec{u} \neq 0} \frac{\|A\vec{u}\|}{\|\vec{u}\|}$$

$\Rightarrow$ For any $\|\vec{u}\|$, $\|A\vec{u}\| \leq \|A\| \cdot \|\vec{u}\|$

Thm If $\|\vec{u}\| = \|\vec{u}\|_1 = \sum |u_i|$, then the induced matrix norm on $A \in \mathbb{C}^{m \times n}$ is:

$$\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{m} |a_{ij}|$$

Proof: $\|A\vec{u}\|_1 = \|\sum_j u_j \vec{a}_j\|_1 \leq \sum_j |u_j| \, \|\vec{a}_j\|_1 \leq \max_j \|\vec{a}_j\|_1 \cdot \sum |u_i|$

$= \max_j \|\vec{a}_j\|_1 \, \|\vec{u}\|_1$

where $A = (\vec{a}_1 \vec{a}_2 \cdots \vec{a}_n)$

$= \|\vec{u}\|_1 \max_j \sum_i |a_{ij}|$

So $\dfrac{\|A\vec{u}\|}{\|\vec{u}\|} \le \max_j \sum_i |a_{ij}|$

But $\vec{u} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow j^{th} \Rightarrow \|A\vec{u}\| \geq \|\vec{a}_j\|$

and therefore $\dfrac{\|A\vec{u}\|}{\|\vec{u}\|} = \max_j \sum_i |a_{ij}|$

Similarly, we have

__Thm__ $\|A\|_\infty = \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}|$

__Proof__ Nearly identical to previous theorem.

Lastly, the most commonly used matrix norm:

__Thm__ $\|A\|_2 = \sqrt{\max_j \lambda_j}$ where λ_j is eigenvalue of A^TA with $A \in \mathbb{R}^{m\times n}$ (or A^*A if $A \in \mathbb{C}^{m\times n}$).

__Proof:__ $\|A\vec{u}\|_2^2 = (A\vec{u}, A\vec{u}) = \vec{u}^T \underbrace{A^TA}_{} \vec{u}$

symmetric positive semi-definite matrix

$\Rightarrow A^TA = PDP^{-1}$, D diagonal

P orthogonal, $P^{-1} = P^T$

$\Rightarrow \max_{\|\vec{u}\|=1} \|A\vec{u}\|_2^2 = \max_{\|\vec{u}\|=1} \|\vec{u}^T PDP^T \vec{u}\|_2^2$

$= \max_{\|\vec{y}\|=1} \|\vec{y}^T D \vec{y}\|_2^2 =$ largest eigenvalue of A^TA.

homework exercise.

Condition number of a problem

$\Rightarrow$ The sensitivity of the "problem" ~~to~~ at the solution.

Ex: For a function $y = f(x)$, how sensitive is y to x? ~~relative estimate is the ...~~ (The "problem" is computing $f(x)$.)

① In an absolute sense:

$$|y - y'| \approx \underbrace{C(x)}_{\text{absolute condition number}} |x - x'|$$

$$\Rightarrow C(x) \approx \frac{|y - y'|}{|x - x'|}$$

$$= \frac{|f(x) - f(x')|}{|x - x'|}$$

for x' close to x,

$$C(x) \approx |f'(x)|.$$

② In a relative sense:

$\frac{|x - x'|}{|x|}$ = relative error

$$\frac{|y' - y|}{|y|} \approx K(x) \left| \frac{x' - x}{x} \right|$$

$$\Rightarrow \quad K(x) \approx \left| \frac{y' - y}{y} \right| \left| \frac{x}{x' - x} \right|$$

$$= \left| \frac{f(x') - f(x)}{x' - x} \right| \left| \frac{x}{y} \right| = \left| \frac{x f'(x)}{f(x)} \right|$$

[redacted]

Ex: Let $y = x^{\frac{1}{3}} = f'(x)$

$$C(x) = \frac{1}{3} x^{-2/3}$$

$$K(x) = \frac{x f'(x)}{f(x)} = \frac{x \cdot \frac{1}{3} \cdot x^{-2/3}}{x^{1/3}} = \frac{\frac{1}{3} x^{1/3}}{x^{1/3}} = \frac{1}{3}$$

$C(x) = \frac{1}{3} \frac{1}{x^{2/3}} < \infty$ for ~~x~~ x away from 0

$K(x)$ is small everywhere

$$f(x+\epsilon) \approx f(x) + f'(x)\epsilon$$

$$\approx f(x) + \frac{1}{3} x^{-2/3} \epsilon$$

$$\Rightarrow |f(x+\epsilon) - f(x)| \approx \underbrace{\frac{1}{3 x^{\frac{2}{3}}}}_{\text{large for } x \text{ near } 0.} \epsilon$$