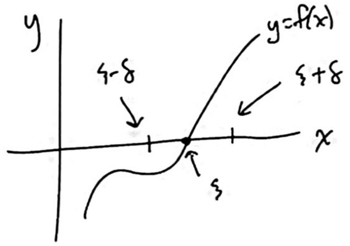
Honors Numerical Analysis

$\boxed{\text{Lecture 4}}$

Theorem (1.8 from Suli & Mayers)

Suppose that f is twice continuously differentiable (i.e. f, f', and f'' are continuous) on the interval $I_\delta = [\xi-\delta, \xi+\delta]$, $\delta > 0$, and that $f(\xi) = 0$ and $f''(\xi) \neq 0$.



Also assume that there exists $A > 0$ such that $\left| \frac{f''(x)}{f'(y)} \right| \leq A$ for all $x, y \in I_\delta$.

If $|\xi - x_0| \leq h$, with $h \leq \min(\delta, \frac{1}{A})$, then the sequence $\{x_k\}$ defined by Newton's Method $x_{k+1} = x_k - f(x_k)/f'(x_k)$, (with starting guess x_0) converges quadratically to ξ.

Proof By Taylor's Theorem,

$\eta_k \in [\xi, x_0] \subseteq I_\delta$

$$f(\xi) = 0 = f(x_0) + f'(x_0)(\xi - x_0) + \frac{f''(\eta_0)}{2}(\xi - x_k)^2.$$

And since by Newton's Method $\xi - x_1 = \xi - x_0 + \frac{f(x_0)}{f'(x_0)}$,

$$(*) \quad \Rightarrow \quad \xi - x_1 = -\frac{f''(\eta_0)}{2f'(x_0)}(\xi - x_0)^2.$$

And therefore

$$|\xi - x_1| \leq \frac{1}{2}\left|\frac{f''(\eta_0)}{f'(x_0)}\right| |\xi - x_0|^2$$

$$\leq \frac{1}{2} A\, |\xi - x_0| \cdot |\xi - x_0|$$

$$\leq \frac{1}{2} A \frac{1}{A} h = \frac{1}{2} h.$$

And once again, $|\xi - x_1| \leq h$, so $\Rightarrow |\xi - x_2| \leq \frac{1}{2^2} h$.

Repeating k times we have that

$$|\xi - x_k| \le \frac{1}{2^k} h \quad \Rightarrow \quad \lim_{k\to\infty} x_k = \xi$$

$$\Rightarrow \boxed{\text{convergence}}$$

Furthermore,

since $$|\xi - x_{k+1}| = \frac{1}{2}\left|\frac{f''(\eta_k)}{f'(x_k)}\right| |\xi - x_k|^2$$

we have that

$$\lim_{k\to\infty} \frac{|\xi - x_{k+1}|}{|\xi - x_k|^2} = \lim_{k\to\infty} \frac{1}{2}\left|\frac{f''(\eta_k)}{f'(x_k)}\right| = \frac{1}{2}\left|\frac{f''(\xi)}{f'(\xi)}\right|$$

since $\lim_{k\to\infty} \eta_k = \xi$, since $\eta_k \in [\xi, x_k]$ in Taylor's Thm.

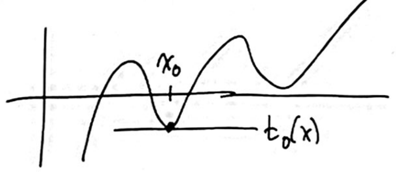
This proves <u>Quadratic Convergence</u>. $\square$

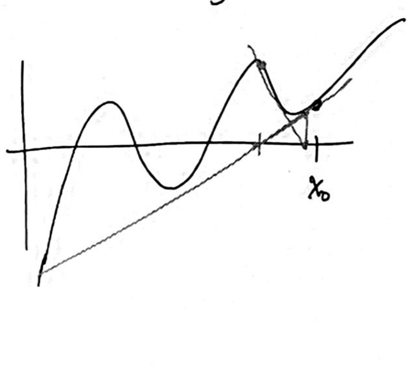
Failures of Newton's Method:

① When Newton's Method converges, and <u>fails</u> to converge, it is usually because $f' = 0$ at the root (or possibly higher derivatives as well).

In this case, the quantity $\frac{f''(x)}{f'(y)}$ may not remain bounded in I_δ

② For some initial guesses, Newton's Method may fail to converge at all.





Rates of Convergence

For the convergent sequence $e_k = |x_k - x^*|$, (i.e. bisection, secant, Newton errors)

we can consider the following limit:

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^{\alpha}} = \mu$$

Bisection

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k} = \lim_{k\to\infty} \frac{L/2^{k+2}}{L/2^{k+1}} = \frac{1}{2}$$

Newton

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^2} = \frac{1}{2}\left|\frac{f''(\xi)}{f'(\xi)}\right|$$

when $f(\xi) = 0$.

Secant (not shown, exercise)

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^{1.62\ldots}} = C$$

When $\alpha = 1 \Rightarrow$ <u>linear</u> convergence (obviously $\mu < 1$)

$\alpha = 2 \Rightarrow$ <u>quadratic</u> convergence

For $\alpha = 1$, we can further ~~determine~~ define the <u>order of convergence</u>.

For $\lim_{k\to\infty} \frac{e_{k+1}}{e_k} = \mu < 1$, set $\rho = -\log_{10}\mu$

ρ is defined to be the <u>asymptotic rate of convergence</u>.

(Can be applied to <u>any</u> sequence.)

<u>Ex</u>: if $\mu = 1/10$, then this means that

$\frac{e_{k+1}}{e_k} \approx 1/10$ and the error goes down by 10% every iteration $\Rightarrow$ x_k obtains <u>one</u> <u>more</u> <u>correct</u> digit every iteration.

$$\rho = -\log_{10}\mu = -\log_{10} 1/10 = -(-1) = 1.$$

= # correct decimal digits gained on successive iterations

<u>Ex</u>: Let $x_k = 1 + 1/10^k$

$$\lim x_k = 1$$

$$e_k = |x_k - 1|$$

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k} = \lim_k \frac{1 + 1/10^{k+1} - 1}{1 + 1/10^k - 1} = \frac{1}{10} = \mu$$

$$\rho = -\log_{10} 1/10 = 1.$$

$\boxed{\text{Note: } \rho \text{ is only defined for } \alpha = 1}$

Solving nonlinear systems

A nonlinear system of equations in n variables $x_1, \dots, x_n$ is given by:

$$\left.\begin{aligned} f_1(x_1,\dots,x_n) &= 0 \\ f_2(x_1,\dots,x_n) &= 0 \\ \vdots \quad & \quad \vdots \\ f_n(x_1,\dots,x_n) &= 0 \end{aligned}\right\}$$

condense into the notation

$$\vec{f}(\vec{x}) = \vec{0}$$

with $\vec{f} = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Once again, there may be 1, 0, or <u>multiple</u> solutions to this system.

- Bisection method does not - in general - extend to higher dimensions since sign changes do not necessarily indicate roots.
- The idea ~~of~~ behind Newton's Method extends directly: Linearize, then solve for an approximate root, linearize again, repeat.