Honors Numerical Analysis

$\boxed{\text{Lecture 3}}$

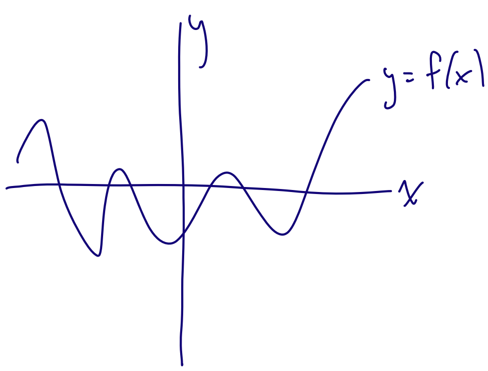
First topic Solving a nonlinear equation

Linear: $3x + 7 = 2$ — Can solve by hand, explicit form of solution

Nonlinear: $\cos x + x^2 - 7 = 5$ — No closed form solution, must use a numerical method

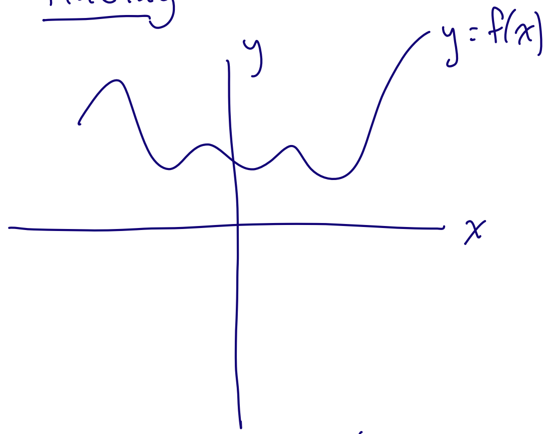
General form of the problem:

Solve $f(x) = 0 \Rightarrow$ Root finding



Many solutions

vs.



No solution (at least if x is required to be real-valued)

Ie. $x^2+1=0 \Rightarrow x = \pm i$

A sufficient condition for a solution to exist on the interval $[a,b]$: $f(a) < 0$ & $f(b) > 0$

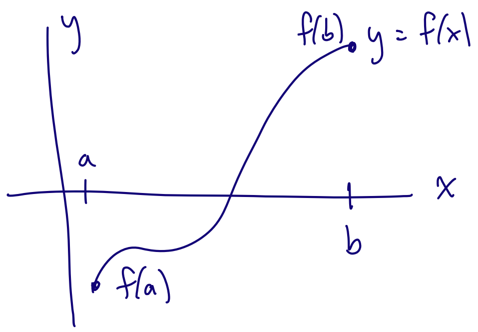
OR $f(a) > 0$ & $f(b) < 0$

Thm: If f is continuous and real-valued on $[a,b]$, and if $f(a)\cdot f(b) < 0$, then there exists an $x \in (a,b)$ s.t. $f(x)=0$.

Proof: Merely apply the Intermediate Valu Thm. (Calc I).

Can we use this Thm to design a numerical method for solving $f(x)=0$?

Bisection



$f(a) < 0$, $f(b) > 0$ $\Rightarrow$ $f(x)=0$ has a solution on $[a,b]$.

Idea: Split the interval in half, apply the same Thm:

If $f\left(\frac{a+b}{2}\right) < 0$, then $f(x)=0$ has a solution on $\left[\frac{a+b}{2}, b\right]$

If $f\left(\frac{a+b}{2}\right) > 0$, then $f(x)=0$ has a solution on $\left[a, \frac{a+b}{2}\right]$.

Split interval in half, and repeat.

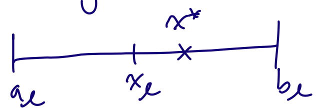
Let $a_0 = a$, $b_0 = b$, the original interval.
$[a_\ell, b_\ell]$ be the interval obtained after ℓ splittings.

Then $b_\ell - a_\ell = \dfrac{b_0 - a_0}{2^\ell} = \dfrac{L}{2^\ell}$ with $L = b_0 - a_0$.

Let $x_\ell = \dfrac{a_\ell + b_\ell}{2}$ be our approximation of the solution to $f(x)=0$ on step ℓ.

When do we stop the splittings? How many steps of bisection do we take?

If we want to guarantee that $|x_\ell - x^*| < \epsilon$ ← some small precision,



↑ true solution, $f(x^*) = 0$.

then we need to choose ℓ such that

$$|x_\ell - x^*| \leq \frac{b_\ell - a_\ell}{2} = \frac{1}{2}\,\frac{b_0 - a_0}{2^\ell} = \frac{1}{2^{\ell+1}} L < \epsilon$$

$$\Rightarrow \quad 2^{\ell+1} > \frac{L}{\epsilon} \qquad \Rightarrow \quad \ell > 1 + \log_2 L/\epsilon .$$

If e_ℓ = error on ℓ^{th} step

$= |x_\ell - x^*|$ = absolute error in x_ℓ.

then $e_{\ell+1} \leq \frac{1}{2} e_\ell$.

$\Rightarrow$ The error goes down by a factor of 2.

This is not very fast.

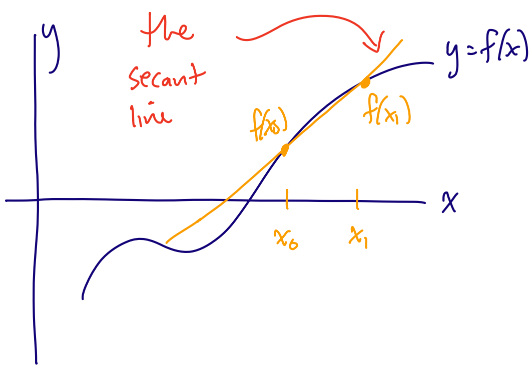
Bisection only used the sign of the function f at a and b.
Can we derive a better (faster) method by using the actual values $f(a)$ and $f(b)$?

The Bisection Method used <u>two</u> pieces of information to approximate the solution to $f(x)=0$ — the sign of the function.

What if we use values? The result is the <u>Secant Method</u>.

Start with two guesses for the root, x_0, x_1:

Graphically:



Find the root of the secant line:

$$s(x) = \frac{f(x_1)-f(x_0)}{x_1 - x_0}(x-x_0) + f(x_0)$$

The root of the secant line satisfies $s(x)=0$:

$$s(x)=0 \Rightarrow x = x_1 - f(x_1)\left(\frac{x_1-x_0}{f(x_1)-f(x_0)}\right)$$

Call this root x_2, the next approximation to the root of f. Also, define $f(x_k) = f_k$.

The secant method generates a sequence of approximations to the root of f by:

$$x_{k+1} = x_k - f_k\left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}}\right).$$

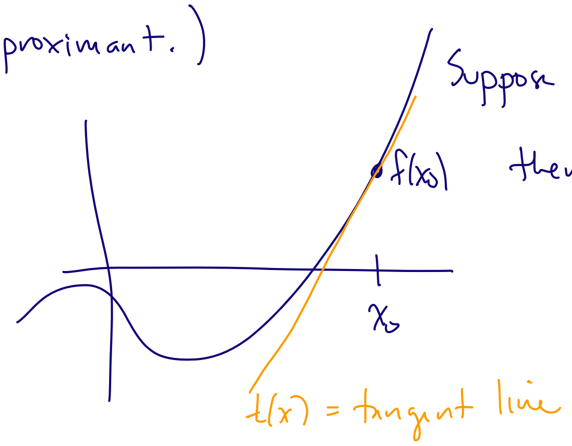
We will re-visit the convergence properties of this method later.

<u>Summary</u> Approximate f by a secant line, find root of secant line, repeat using new approximate root.

Newton's Method

What if we are now allowed to use <u>derivative</u> information to approximate f? (And then find the root of this approximant.)

Suppose we know $f(x_0)$ and $f'(x_0)$, then we can draw the tangent line.



The equation of the tangent line is given as:

$$t(x) = f(x_0) + f'(x_0)(x-x_0)$$

The root of the tangent line satisfies $t(x)=0$

$$\Rightarrow \quad x = x_0 - \frac{f(x_0)}{f'(x_0)} \equiv x_1$$

x_1 is the next approximation to the root of f.

Repeating this procedure at x_1 we obtain Newton's Method:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Alternative interpretation: the tangent line is a first-order Taylor approximation to f.

Recall: The Taylor series of f about x_0 (assuming f has infinite derivatives):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \dots$$

Truncating this series after the first two terms gives:

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0)$$

<u>If</u> x_0 is close to the root of f, ξ, $f(\xi)=0$, then we get that

$$f(\xi) = 0 \approx f(x_0) + f'(x_0)(\xi - x_0)$$

Solve for $\xi \Rightarrow$ $\left.\xi \approx x_0 - \frac{f(x_0)}{f'(x_0)}.\right]$ This is Newton's Method.

$\uparrow$ approximately

<u>Summary</u> The Secant Method and Newton's Method <u>both</u> work by the same mechanism: approximate f by a linear function, find the root of that linear function. Update and repeat.

<u>Convergence Behavior</u>

Let ξ be the true root of f, ie $f(\xi)=0$.

We are interested in how the <u>absolute error</u> of x_k changes from iteration to iteration. $e_k = |\xi - x_k|$.

For the bisection method:

$$e_{k+1} \approx \frac{1}{2} e_k$$

It turns out that Newton converges <u>quadratically</u>:

$$e_{k+1} \approx A e_k^2$$ $\leftarrow$ this is fast!

If $e_0 = 10^{-1}$ $\quad e_3 \sim 10^{-8}$

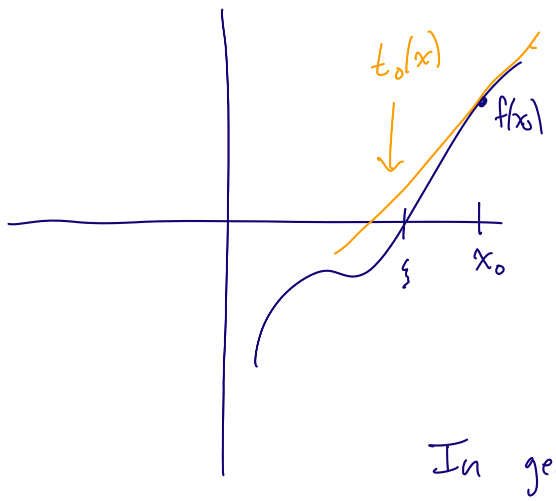
$e_1 \sim 10^{-2}$ $\quad e_4 \sim 10^{-16}$

$e_2 \sim 10^{-4}$

We will prove this rate next class.

Analysis of Newton's Method

Recall: Newton's Method approximates a function by its tangent line and then finds the root of the tangent line. And then repeats this:



$f(\xi) = 0$

$t_0(x) = f(x_0) + f'(x_0)(x - x_0)$

$t_0(x_1) = 0 \implies \underbrace{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}_{\text{Newton iteration}}$

In general, Newton's method is:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Two questions to ask:

1. Does Newton's method converge?

   I.e. Does the sequence $\{x_k\}$ converge to ξ?

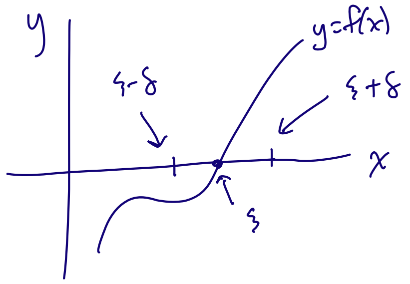
   I.e. Is $\lim_{k \to \infty} x_k = \xi$?

2. If it converges, how fast does it converge.

We answer these questions via Theorem / Proof, placing certain assumptions on f.

Theorem (1.8 from Suli & Mayers)

Suppose that f is twice continuously differentiable (i.e. f, f', and f'' are continuous) on the interval $I_\delta = [\xi-\delta, \xi+\delta]$, $\delta>0$, and that $f(\xi)=0$ and $f''(\xi) \neq 0$.



Also assume that there exists $A>0$ such that $\left|\dfrac{f''(x)}{f'(y)}\right| \leq A$ for all $x, y \in I_\delta$.

If $|\xi - x_0| \leq h$, with $h \leq \min\left(\delta, \frac{1}{A}\right)$, then the sequence $\{x_k\}$ defined by Newton's Method $x_{k+1} = x_k - f(x_k)/f'(x_k)$, (with starting guess x_0) converges quadratically to ξ.

Proof By Taylor's Theorem,

$\eta_k \in [\xi, x_0] \subseteq I_\delta$

$$f(\xi) = 0 = f(x_0) + f'(x_0)(\xi - x_0) + \frac{f''(\eta_0)}{2}(\xi - x_k)^2 .$$

And since by Newton's Method $\xi - x_1 = \xi - x_0 + \dfrac{f(x_0)}{f'(x_0)}$,

$$(*) \quad \Rightarrow \quad \xi - x_1 = -\frac{f''(\eta_0)}{2f'(x_0)}(\xi - x_0)^2$$

And therefore $|\xi - x_1| \leq \dfrac{1}{2}\left|\dfrac{f''(\eta_0)}{f'(x_0)}\right| |\xi - x_0|^2$

$$\leq \frac{1}{2} A \, |\xi - x_0| \cdot |\xi - x_0|$$

$$\leq \frac{1}{2} A \, \frac{1}{A} \, h = \frac{1}{2} h.$$

And once again, $|\xi - x_1| \leq h$, so $\Rightarrow |\xi - x_2| \leq \frac{1}{2^2} h$.

Repeating k times we have that

$$|\xi - x_k| \leq \frac{1}{2^k} h \implies \lim_{k \to \infty} x_k = \xi$$

$$\implies \boxed{\text{convergence}}$$