First topic: Solving a nonlinear equation

Linear: \( 3x + 7 = 2 \)  
Can solve by hand, explicit form of solution

Nonlinear: \( \cos x + x^2 - 7 = 5 \)  
No closed form solution, must use a numerical method

General form of the problem:

Solve \( f(x) = 0 \) \( \Rightarrow \) Root finding

A sufficient condition for a solution to exist on the interval \([a, b]\): \( f(a) < 0 \) \& \( f(b) > 0 \)

or \( f(a) > 0 \) \& \( f(b) < 0 \)
**Theorem:** If $f$ is continuous and real-valued on $[a,b]$, and if $f(a) \cdot f(b) < 0$, then there exists an $x \in (a,b)$ s.t. $f(x) = 0$.

**Proof:** Merely apply the Intermediate Value Thm. (Calc I).

Can we use this Thm to design a numerical method for solving $f(x) = 0$?

**Bisection**

If $f(a) \cdot f(b) < 0$, then $f(x) = 0$ has a solution on $[a,b]$.

**Idea:** Split the interval in half, apply the same Thm:

If $f(\frac{a+b}{2}) < 0$, then $f(x) = 0$ has a solution on $[\frac{a+b}{2}, b]$.

If $f(\frac{a+b}{2}) > 0$, then $f(x) = 0$ has a solution on $[a, \frac{a+b}{2}]$.

Split interval in half, and repeat.

Let $a_0 = a$, $b_0 = b$, the original interval.

$[a_{l-1}, b_{l-1}]$ be the interval obtained after $l$ splittings.

Then $b_l - a_l = \frac{b_0 - a_0}{2^l} = \frac{L}{2^l}$ with $L = b_0 - a_0$.

Let $x_l = \frac{a_l + b_l}{2}$ be our approximation of the solution to $f(x) = 0$ on step $l$.  

\[ x_l = \frac{a_l + b_l}{2} \]
When do we stop the splittings? How many steps of bisection do we take?

If we want to guarantee that \( |x_e - x^*| < \varepsilon \),

then we need to choose \( l \) such that

\[
|x_e - x^*| \leq \frac{b_e - a_e}{2} = \frac{1}{2} \frac{b_0 - a_0}{2^l} = \frac{1}{2^{l+1}} L \leq \varepsilon
\]

\[
\Rightarrow 2^{l+1} > \frac{L}{\varepsilon} \quad \Rightarrow \quad l > l + \log_2 \frac{L}{\varepsilon}.
\]

If \( e_l = \text{error on } l^{th} \text{ step} \)

\[
= |x_e - x^*| = \text{absolute error in } x_e.
\]

then \( e_{l+1} \leq \frac{1}{2} e_l \).

\[
\Rightarrow \text{The error goes down by a factor of 2.}
\]

This is not very fast.

Bisection only used the sign of the function \( f \) at \( a \) and \( b \).
Can we derive a better (faster) method by using the actual values \( f(a) \) and \( f(b) \)?
The Bisection Method used two pieces of information to approximate the solution to \( f(x) = 0 \) — the sign of the function.

What if we use values? The result is the **Secant Method**.

Start with two guesses for the root, \( x_0, x_1 \).

Graphically:

![Graph showing secant method](image)

Find the root of the secant line:

\[
s(x) = \frac{f(x_1)f(x_0)}{f(x_1) - f(x_0)} (x - x_0) + f(x_0)
\]

The root of the secant line satisfies \( s(x) = 0 \):

\[
s(x) = 0 \Rightarrow x = x_1 - \frac{f(x_1)}{f(x_1) - f(x_0)} (x_1 - x_0)
\]

Call this root \( x_2 \), the next approximation to the root of \( f \). Also, define \( f(x_2) = f_x \).

The secant method generates a sequence of approximations to the root of \( f \) by:

\[
x_{k+1} = x_k - f_x \left( \frac{x_k - x_{k-1}}{f_x - f_{x-1}} \right).
\]

We will revisit the convergence properties of this method later.

**Summary** Approximate \( f \) by a secant line, find root of secant line, repeat using new approximate root.
Newton's Method

What if we are now allowed to use derivative information to approximate $f$? (And then find the root of this approximant.)

Suppose we know $f(x_0)$ and $f'(x_0)$, then we can draw the tangent line.

![Tangent Line Diagram]

The equation of the tangent line is given as:

$$ t(x) = f(x_0) + f'(x_0)(x-x_0) $$

The root of the tangent line satisfies $t(x) = 0$

$$ \Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)} = x_1 $$

$x_1$ is the next approximation to the root of $f$.

Repeating this procedure at $x_1$, we obtain Newton's Method:

$$ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} $$

Alternative interpretation: the tangent line is a first-order Taylor approximation to $f$.

Recall: The Taylor series of $f$ about $x_0$ (assuming $f$ has infinite derivatives):

$$ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n $$

$$ = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!} (x-x_0)^2 + \frac{f'''(x_0)}{3!} (x-x_0)^3 + \ldots $$
Truncating this series after the first two terms gives:
\[ f(x) \approx f(x_0) + f'(x_0)(x-x_0) \]

If \( x_0 \) is close to the root of \( f \), i.e., \( f(x) = 0 \), then we get that
\[ f(x) = 0 \approx f(x_0) + f'(x_0)(x-x_0) \]

Solve for \( x \Rightarrow x \approx x_0 - \frac{f(x_0)}{f'(x_0)} \). This is Newton's Method.

**Summary**
The Secant Method and Newton's Method both work by the same mechanism: approximate \( f \) by a linear function, find the root of that linear function.

Update and repeat.

**Convergence Behaviour**
Let \( s \) be the true root of \( f \), i.e. \( f(s) = 0 \).
We are interested in how the absolute error of \( x_k \) changes from iteration to iteration, \( e_k = |s-x_k| \).

For the bisector method:

\[ e_{k+1} \approx \frac{1}{2} e_k \]

It turns out that Newton converges quadratically:

\[ e_{k+1} \approx A e_k^2 \]  

If \( e_0 = 10^{-1} \)
\[ e_1 \sim 10^{-2} \]
\[ e_2 \sim 10^{-4} \]
\[ e_3 \sim 10^{-8} \]
\[ e_4 \sim 10^{-16} \]

We will prove this rate next class.
Analysis of Newton's Method

Recall: Newton's Method approximates a function by its tangent line and then finds the root of the tangent line. And then repeats this:

\[ f(s) = 0 \]
\[ t_0(x) = f(x_0) + f'(x_0)(x-x_0) \]
\[ t_0(x_1) = 0 \implies x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \]

Newton iteration

In general, Newton's method is:

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \]

Two questions to ask:

1. Does Newton's method converge?
   - I.e., Does the sequence \( \{x_n\} \) converge to \( s \)?
   - I.e., Is \( \lim_{k \to \infty} x_k = s \)?

2. If it converges, how fast does it converge?

We answer these questions via Theorem / Proof, placing certain assumptions on \( f \).
**Theorem (1.8 from Suli & Mayers)**

Suppose that \( f \) is twice continuously differentiable (i.e., \( f, f', \) and \( f'' \) are continuous) on the interval \( I_\delta = [s - \delta, s + \delta] \), \( \delta > 0 \), and that \( f(s) = 0 \) and \( f''(s) \neq 0 \).

![Graph of function](image)

Also assume that there exists \( A > 0 \) such that
\[
\left| \frac{f''(x)}{f'(y)} \right| \leq A \text{ for all } x, y \in I_\delta.
\]

If \( |s - x_0| \leq h \), with \( h \leq \min \left( \delta, \frac{1}{A} \right) \), then the sequence \( \{x_k\} \) defined by Newton's Method
\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},
\]
(with starting guess \( x_0 \)) converges quadratically to \( s \).

**Proof** By Taylor's Theorem,
\[
f(s) = 0 = f(x_0) + f'(x_0)(s - x_0) + \frac{f''(\xi)}{2}(s - x_0)^2.
\]

And since by Newton's Method
\[
s - x_1 = s - x_0 + \frac{f(x_0)}{f'(x_0)} - \frac{f''(\xi)}{2f'(x_0)}(s - x_0)^2
\]

\( \Rightarrow \)
\[
s - x_1 = -\frac{f''(\xi)}{2f'(x_0)}(s - x_0)^2
\]

And therefore
\[
|s - x_1| \leq \frac{1}{2} \left| \frac{f''(\xi)}{f'(x_0)} \right| |s - x_0|^2
\]

\[
\leq \frac{1}{2} A |s - x_0| \cdot |s - x_0|
\]

\[
\leq \frac{1}{2} A \cdot \frac{1}{A} h = \frac{1}{2} h.
\]

And once again, \( |s - x_1| \leq h \), so \( \Rightarrow \)
\[
|s - x_2| \leq \frac{1}{2} h.
\]
Repeating $k$ times we have that

$$|s - x_k| \leq \frac{1}{2^k} h \implies \lim_{k \to \infty} x_k = s$$

\(\iff \) convergence