Homework 3

Due: Friday 1:30pm, March 13, 2020, **in recitation**

Notes on the first (and all subsequent) assignments:

Submission: Homework assignments must be submitted in the class on the due date. If you cannot attend the class, please send your solution per email as a single PDF before class. Please hand in cleanly handwritten or typed (preferably with LaTeX) homework. Feel free to use original homework LaTeX document to write-up your homework. If you are required to hand in code or code listings, this will explicitly be stated on that homework assignment.

Collaboration: NYU's integrity policies will be enforced. You are encouraged to discuss the problems with other students. However, you must write (i.e. type) every line of code yourself and also write up your solutions independently. Copying of any portion of someone else's solution/code or allowing others to copy your solution/code is considered cheating.

Plotting and formatting: Plot figures carefully and think about what you want to illustrate with a plot. Choose proper ranges and scales (e.g. semilogx, semilogy, loglog), always label axes, and give meaningful titles. Sometimes, using a table can be useful, but never submit pages of numbers. Discuss what we can observe in, and learn from, a plot. If you do print numbers, in MATLAB for example, use `fprintf` to format the output nicely. Use `format compact` and other format commands to control how MATLAB prints things. When you create figures using MATLAB (or Python or Julia), please try to export them in a vector graphics format (.eps, .pdf, .dxf) rather than raster graphics or bitmaps (.jpg, .png, .gif, .tif). Vector graphics-based plots avoid pixelation and thus look much cleaner.

Programming: This is an essential part of this class. We will use MATLAB for demonstration purposes in class, but you are free to use other languages. The TA will give an introduction to MATLAB in the first few recitation classes. Please use meaningful variable names, try to write clean, concise and easy-to-read code. As detailed in the syllabus, in order to receive full credit, your code must be thoroughly commented.

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1. [**10 pts**] Let the matrix $\mathbf{A}$ be given as:

$$\mathbf{A} = \begin{pmatrix} -8 & -12 & -12 & -7 \\ -4 & -3 & -2 & -1 \\ -8 & -9 & -6 & -3 \\ -8 & -12 & -10 & -5 \end{pmatrix}.$$

   Find the **LU** decomposition of $\mathbf{A}$, allowing for row pivoting. Your final factorization will be of the form $\mathbf{PA} = \mathbf{LU}$, where $\mathbf{P}$ is a permutation matrix. You must show all your work to obtain full credit.

2. (a) [**5 pts**] Let $\mathbf{v} = (v_1 \cdots v_n)^T \in \mathbb{R}^n$. Prove the following two inequalities:

$$\|\mathbf{v}\|_\infty \le \|\mathbf{v}\|_2,$$
$$\|\mathbf{v}\|_2^2 \le \|\mathbf{v}\|_1 \, \|\mathbf{v}\|_\infty$$

   For each inequality, give an example of a non-zero vector $\mathbf{v}$ for which equality is obtained. Finally, show that

$$\|\mathbf{v}\|_\infty \le \|\mathbf{v}\|_2 \le \|\mathbf{v}\|_1,$$
$$\|\mathbf{v}\|_2 \le \sqrt{n}\|\mathbf{v}\|_\infty.$$

(b) [**5 pts**] Next, let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be any real-valued matix. Show that

$$\|\mathbf{A}\|_\infty \leq \sqrt{n}\|\mathbf{A}\|_2,$$
$$\|\mathbf{A}\|_2 \leq \sqrt{m}\|\mathbf{A}\|_\infty.$$

For each inequality, give an example matrix $\mathbf{A}$ for which equality is attained.

3. Another commonly used matrix norm in linear algebra is the *Frobenius norm*, defined for a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ as:

$$\|\mathbf{A}\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}. \tag{1}$$

(a) [**4 pts**] Prove that $\|\mathbf{A}\|_F = \sqrt{\text{trace}\left(\mathbf{A}^T\mathbf{A}\right)}$.

(b) [**3 pts**] Prove also that $\|\mathbf{A}\|_F = \sqrt{\sum_j \sigma_j^2}$, where σ_j are the singular values of $\mathbf{A}$.

(c) [**3 pts**] Show that $\|\mathbf{A}\|_2 \leq \|\mathbf{A}\|_F$.

4. [**10 pts**] Prove that, for any nonsingular matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$,

$$\kappa_2(\mathbf{A}) = \sqrt{\frac{\lambda_n}{\lambda_1}},$$

where λ_1 is the smallest and λ_n is the largest eigenvalue of the matrix $\mathbf{A}^T\mathbf{A}$.

5. Consider the lower-triangular matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

(a) [**4 pts**] Compute the ℓ_1 norm of the matrix and its inverse, i.e. $\|\mathbf{A}\|_1$ and $\|\mathbf{A}^{-1}\|_1$. What is $\kappa_1(\mathbf{A})$?

(b) [**1 pt**] Compute $\mathbf{A}^T\mathbf{A}$.

(c) [**1 pt**] Show that any vector $\mathbf{x} \neq \mathbf{0}$ is an eigenvector of $\mathbf{A}^T\mathbf{A}$ with eigenvalue $\lambda = 1$ provided that $x_1 = 0$ and $x_2 + \cdots x_n = 0$.

(d) [**1 pt**] Show also that there are two eigenvectors with $x_2 = \cdots = x_n$, and find the corresponding eigenvalues.

(e) [**3 pts**] Finally, show that the ℓ_2 condition number is given by:

$$\kappa_2(\mathbf{A}) = \frac{n+1}{2}\left(1 + \sqrt{1 - \frac{4}{(n+1)^2}}\right).$$