April 30, 2020

Numerical Analysis

Last time: Initial Value problems:

$$y'(t) = f(t, y)$$
$$y(t_0) = y_0$$

Formal solution: $$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau$$

Finite Differences:

y' can be approximated as:

$$y'(t) \approx \frac{y(t+h) - y(t)}{h}$$

Insert into initial value problem:

$$\frac{y(t+h) - y(t)}{h} \approx f(t, y(t))$$

$$y(t+h) \approx y(t) + h \cdot f(t, y(t)).$$ Forward Euler's Method

Error analysis of the finite difference approximation:

$$\left| y'(t) - \frac{y(t+h) - y(t)}{h} \right| \leq \underbrace{\mathcal{O}(h)}_{\text{truncation error}} + \underbrace{\mathcal{O}\left(\frac{\epsilon}{h}\right)}_{\text{round-off error}}$$

machine precision

① h must be chosen to balance these errors.

② Higher-order approximations can be obtained using Richardson Extrapolation.

General use of Richardson Extrapolation:

If some quantity L has truncation error (or order of convergence):

$$L = \underbrace{\varphi_0(h)}_{\text{previous example}} + a_1 h + a_2 h^2 + a_3 h^3 + \ldots$$

this was
$$\varphi_0(h) = \frac{y(t+h) - y(t-h)}{2h}$$

$$L = \varphi_0\left(\frac{h}{2}\right) + a_1 \frac{h}{2} + \frac{a_2}{4} h^2 + \frac{a_3}{8} h^3 + \ldots$$

$\Rightarrow$ that $2\varphi_0\left(\frac{h}{2}\right) - \varphi_0(h) = L + O(h^2)$.

Ex: If I want to apply Richardson Extrapolation to evaluate $y'(t) = L$

$$\frac{y(t+h) - y(t)}{h} = y'(t) + O(h)$$
$$= y'(t) + ah + bh^2 + \ldots$$

Rewrite as: $$y'(t) = \underbrace{\frac{y(t+h) - y(t)}{h}}_{\varphi_0(h)} + ah + bh^2 + \ldots$$

$\Rightarrow$ $2\varphi_0(h/2) - \varphi_0(h) = y'(t) + O(h^2)$.

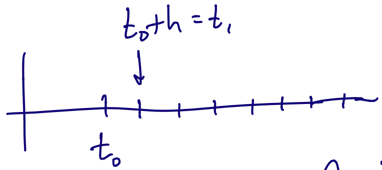
Depending on the order of approximation of φ_0 to L, the actual coefficients in Richardson will differ. (compare with last lecture.)

This analysis does not include any round-off effects. (More detailed analysis is required.)

Back to initial value problems: Error analysis, local vs. global.

Consider $$y'(t) = \frac{y(t+h) - y(t)}{h} + \frac{h}{2} y''(\xi) \quad (*)$$ $\xi \in [t, t+h]$ by Taylor's Theorem.

Let $t_k = t_0 + kh$



Goal: Compute approximate solution to IVP at each t_k.

One step of Forward Euler:

$$y(t_1) \approx y(t_0) + h \cdot f(t_0, y(t_0))$$
$$= y_0 + h \cdot f(t_0, y_0).$$

<u>Local Error</u>: The true IVP can be written as:

$$y'(t) = f(t, y(t)) \quad \underset{\text{insert } (*)}{\Rightarrow} \quad \frac{y(t+h) - y(t)}{h} = f(t, y(t)) + \frac{h}{2} y''(\xi)$$

When the initial value problem is in this form, the <u>local error</u> is

First order.

Some notation going forward:

$t_k = t_0 + k \cdot h$

$y(t_k)$ = <u>true</u> <u>value</u> of y at t_k.

y_k = approximate value of $y(t_k)$ obtained via numerical solution of the IVP.

Forward Euler can then be written as:

$$\underbrace{\frac{y_{k+1} - y_k}{h}}_{\approx y'(t_k)} = \underbrace{f(t_k, y_k)}_{\text{approx to } f(t_k, y(t_k))} \Rightarrow y_{k+1} = y_k + h \cdot f(t_k, y_k).$$

Definition Consistency

Euler's Method is consistent:

$$\lim_{h\to 0} \frac{y(t_{k+1}) - y(t_k)}{h} = y'(t_k)$$

$$= f(t_k, y(t_k))$$

" as $h \to 0$, $\frac{y_{k+1} - y_k}{h} = f(t_k, y_k) \longrightarrow y'(t) = f(t, y(t))$.

Global Error on interval $[t_0, T]$

$$= \max_{t_k \in [t_0, T]} |y(t_k) - y_k|$$

Maximum error on the entire interval on which you are solving $y' = f$.

Euler: Global error is $O(h)$.

Proof: Euler's Method : $y_{k+1} = y_k + h f(t_k, y_k)$

Taylor's Theorem : $y(t_{k+1}) = y(t_k) + h \cdot y'(t_k) + \frac{h^2}{2} y''(\xi_k)$

$$= y(t_k) + h f(t_k, y(t_k)) + \frac{h^2}{2} y''(\xi_k).$$

Taylor − Euler = $d_{k+1} = y(t_{k+1}) - y_k$

$$= d_k + h\left(f(t_k, y(t_k)) - \underbrace{f_k}_{=f(t_k,y_k)}\right) + \frac{h^2}{2} y''(\xi_k).$$

Taking absolute values:

$$|d_{k+1}| \le |d_k| + h\,|f(t_k, y(t_k)) - f_k| + \frac{h^2}{2}|y''(\xi_k)|$$

$$\le |d_k| + h\,\underbrace{L\,|d_k|}_{\text{Lipschitz in } y} + \frac{h^2}{2} M$$

$M = \max |y''|$ over $[t_k, t_{k+1}]$

Lipschitz in y $|f(t,x) - f(t,y)| \le L|x-y|$

$$= (1 + hL)|d_k| + \frac{h^2}{2} M$$

Lemma If $\gamma_{k+1} \le (1+\alpha)\gamma_k + \beta$

then $\gamma_n \le e^{n\alpha}\gamma_0 + \dfrac{e^{n\alpha}-1}{\alpha}\beta$

$\boxed{\text{Proof by induction}}$

$\Rightarrow$ this implies that

$$|d_k| \le e^{khL}|d_0| + \frac{e^{khL}-1}{hL}\frac{h^2}{2}M$$

$$= e^{khL}|d_0| + \frac{e^{khL}-1}{L}\frac{h}{2}M.$$

Taking maximum over $t_k \in [t_0, T]$

$M = \max_{[t_0,T]} y''(t)$

$$\max_k |d_k| \le e^{L(T-t_0)}|d_0| + \underbrace{\frac{e^{L(T-t_0)}-1}{L}}_{\text{constant}}\,\frac{h}{2}\,\underbrace{M}_{\text{constant}}$$

$=0$ since $d_0 = |y(t_0) - y_0|$

$$= Ch \sim \mathcal{O}(h).$$

For forward Euler, both the local and global errors were first order.

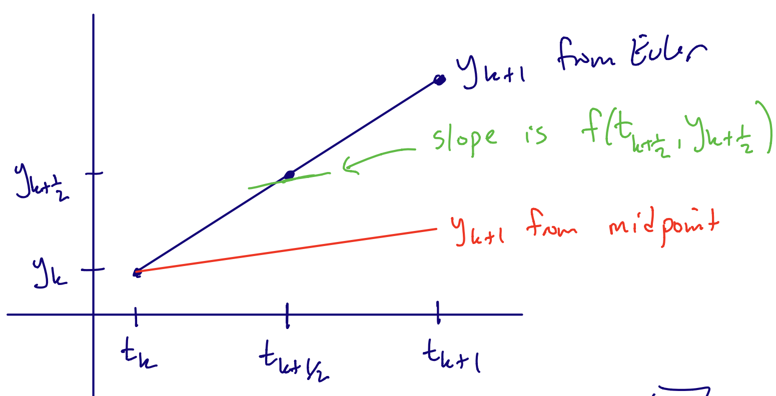
Other Schemes for solving Initial Value Problems

Midpoint Method

"half step of forward Euler, then a full step"

$$y_{k+\frac{1}{2}} = y_k + \frac{h}{2} f(t_k, y_k)$$

$$y_{k+1} = y_k + h f\left(t_{k+\frac{1}{2}}, y_{k+\frac{1}{2}}\right)$$



Compare Midpoint method with two half Euler steps:

$$y_{k+\frac{1}{2}} = y_k + \frac{h}{2} f_k$$

$$y_{k+1} = y_{k+\frac{1}{2}} + \frac{h}{2} f_{k+\frac{1}{2}}$$

$$= y_k + \frac{h}{2} f_k + \frac{h}{2} f_{k+1/2}$$

$$= y_k + h \left(\frac{f_k + f_{k+1/2}}{2} \right)$$

Different than the midpoint method.

For the midpoint method, the Local Truncation error is $O(h^2)$.

Quadrature Method Rules

$$y'(t) = f(t, y(t))$$

$$y(t_0) = y_0$$

Solution: $y(t) = y_0 + \underbrace{\int_{t_0}^{t} f(\tau, y(\tau))\, d\tau}_{\text{use any quadrature rule to approximate this integral.}}$

Trapezoidal Method

$$y(t+h) = y(t) + \int_{t}^{t+h} f(s, y(s))\, ds$$

$$\approx y(t) + \frac{h}{2} \Big(f(t, y(t)) + f(t+h, y(t+h)) \Big)$$

$$\Rightarrow \quad y_{k+1} = y_k + \frac{h}{2} \Big(f(t_k, y_k) + f(t_{k+1}, y_{k+1}) \Big)$$

This equation must be solved for y_{k+1} $\Rightarrow$ Implicit method

Euler and Midpoint are Explicit Methods.