April 16, 2020

Numerical Analysis

Function approximation in the 2-norm.

Goal For a given function f on $[a,b]$, find $p_n \in P_n$ that $\| f - p_n \|_2$ is as small as possible.

$\Rightarrow$ We previously derived that this is a least squares problem and therefore there exists a constructive solution, i.e.

If $\quad p_n(x) = c_0 \varphi_0(x) + c_1 \varphi_1(x) + \ldots + c_n \varphi_n(x)$

form a basis for P_n, then the coefficients c_j are obtained by solving a linear system.

Finally, if the functions φ_j are orthonormal, then the coefficients c_j are merely inner products:

$$c_j = (f, \varphi_j).$$

(analogous to Gram-Schmidt, or orthogonal projections).

This approximation of f is equivalent to finding its orthogonal projection onto P_n under the inner product:

$$(f, g) = \int_a^b f(x)\, g(x)\, w(x)\, dx.$$

Definition If the sequence of polynomials $\varphi_0, \varphi_1, \ldots, \varphi_n$, with $\deg \varphi_j = j$, on the interval (a,b) satisfies

$$\int_a^b \varphi_j(x)\, \varphi_k(x) = 0 \quad \text{if } j \neq k,$$

then $\varphi_0, \ldots, \varphi_n$ is a system of orthogonal polynomials (with weight $w(x) = 1$).

(Likewise we could define $(f,g) = \int_a^b f(x)\, g(x)\, w(x)\, dx$.)

Example: Find $\varphi_0, \varphi_1, \varphi_2$ on $[-1,1]$ with weight function $w(x) = 1$.

Set $\varphi_0(x) = 1$.

$\varphi_1(x) = ax + b$.

If $(\varphi_0, \varphi_1) = 0$ then $\int_{-1}^{1} 1 \cdot (ax+b)\, dx = 0$

$2b = 0 \Rightarrow b = 0$.

Set $\varphi_1(x) = x$.

Let $\varphi_2(x) = x^2 + bx + c$

Two conditions must be satisfied:

$$\int_{-1}^{1} \varphi_0(x)\varphi_2(x)\, dx = 0 \qquad \int_{-1}^{1} \varphi_1(x)\varphi_2(x)\, dx = 0$$

$$\int_{-1}^{1} (x^2+bx+c)\, dx = 0 \qquad \int_{-1}^{1} x(x^2+bx+c)\, dx = 0$$

$$\frac{2}{3} + 2c = 0 \qquad \frac{2}{3}b = 0 \Rightarrow b = 0.$$

$$c = \frac{-1}{3}$$

So we set $\varphi_2(x) = x^2 - \frac{1}{3}$.

So by construction, 1, x, and $x^2 - 1/3$ are orthogonal on $[-1,1]$.

We could do the same calculations for $\varphi_3, \varphi_4, \ldots$

The resulting polynomials are known as Legendre Polynomials. They form an orthogonal basis for all of $L^2[-1,1]$ under the inner product $(f,g) = \int_{-1}^{1} f(x)g(x)\, dx$.

$f \in L^2[-1,1]$ iff $\underbrace{\int_{-1}^{1} (f(x))^2\, dx < \infty}_{\|f\|_2^2 < \infty}$.

Legendre polynomials can also be constructed another way as well: the Gram-Schmidt process.

Start with $P_0(x) = 1$, $P_1(x) = x$. $\Big]$ automatically orthogonal.

Set $m_2(x) = x^2$. $\leftarrow$ linearly independent from P_0, P_1.

For Gram-Schmidt: $P_2 = m_2 - \operatorname{Proj}_{\{P_0,P_1\}} m_2$

$$= m_2 - \frac{(m_2, P_0)}{(P_0, P_0)} P_0 - \frac{(m_2, P_1)}{(P_1, P_1)} P_1$$

So $P_2(x) = x^2 - \frac{2}{3}\frac{1}{2} \cdot 1 - 0$

$= x^2 - \frac{1}{3}$ Exactly the same as before.

So in general, compute

$$P_n = m_n - \sum_{l=0}^{n-1} \frac{(m_n, P_l)}{(P_l, P_l)} P_l$$

$$P_n(x) = x^n - \sum_{l=0}^{n-1} \frac{P_l(x)}{\|P_l\|_2^2} \int_{-1}^{1} x^n P_l(x)\, dx$$

These polynomials can be scaled to <u>any</u> interval.

If $\int_{-1}^{1} P_j(x) P_k(x)\, dx = 0$ if $j \neq k$, then it is easy

to show that $\int_a^b P_j(t) P_k(t)\, dt = 0$ if $j \neq k$

where $t = \left(\frac{x+1}{2}\right)(b-a) + a$.

<u>Ex</u>: Chebyshev Polynomials

We know that $\int_0^\pi \cos mt \cos nt\, dt = 0$ if $m \neq n$.

Let $t = a\cos x$, $dt = \frac{1}{\sqrt{1-x^2}}\, dx$, $t: -1 \to 1$

$$\Rightarrow \int_{-1}^{1} \underbrace{\cos(m \operatorname{acos} x)}_{T_m(x)} \cdot \underbrace{\cos(n \operatorname{acos} x)}_{T_n(x)} \cdot \frac{dx}{\sqrt{1-x^2}} = 0 \quad \text{if } m \neq n.$$

$$\Rightarrow \int_{-1}^{1} T_m(x)\, T_n(x) \frac{1}{\sqrt{1-x^2}}\, dx = 0 \quad \text{if} \quad m \neq n.$$

Therefore, the functions $T_0, T_1, T_2, \ldots$ are orthogonal on $[-1,1]$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$.

<u>Theorem</u> If $\int_a^b |f(x)|^2 w(x)\, dx < \infty$ (ie. $f \in L_w^2[a,b]$), there is a <u>unique</u> degree n polynomial p_n such that

$$\| f - p_n \|_{w,2} = \min_{q \in P_n} \| f - q \|_{w,2}$$

where $\| f \|_{w,2}^2 = \int_a^b |f(x)|^2 w(x)\, dx.$

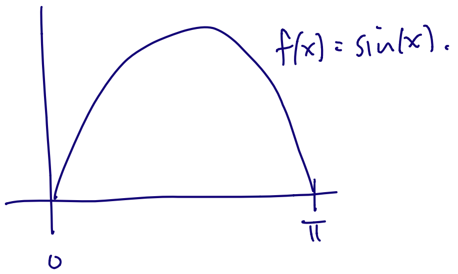
<u>Proof</u>: Gram-Schmidt, solve directly for the coefficients of the associated orthogonal polynomial expansion to compute the approximation.

Note: This is just linear algebra!

Famous sets of orthogonal polynomials:

$w(x)$	(a,b)	Polynomial
1	$(-1,1)$	Legendre
$1/\sqrt{1-x^2}$	$(-1,1)$	Chebyshev
e^{-x}	$(0,\infty)$	Laguerre
e^{-x^2}	$(-\infty,\infty)$	Hermite.

Example Compute the best quadratic 2-norm approximation to $f(x) = \sin x$ on $(0, \pi)$ with weight function $w(x) = 1$.



The first 3 Legendre polynomials on $(-1, 1)$ are:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = x^2 - \frac{1}{3}$$

To define these functions on $(0, \pi)$, let $t = \frac{x+1}{2}\pi$

$$\Rightarrow \quad x = \frac{2t}{\pi} - 1$$

So the shifted orthogonal polynomials are:

$$\tilde{P}_0(t) = 1$$

$$\tilde{P}_1(t) = x = \frac{2}{\pi}t - 1 = \frac{2}{\pi}\left(t - \frac{\pi}{2}\right)$$

$$\tilde{P}_2(t) = x^2 - \frac{1}{3} = \left(\frac{2t}{\pi} - 1\right)^2 - \frac{1}{3}$$

$$= \frac{4t^2}{\pi^2} - \frac{4t}{\pi} + 1 - \frac{1}{3}$$

$$= \frac{4}{\pi}\left(\frac{t^2}{\pi} - t + \frac{\pi}{6}\right).$$

The best approximation is given by the projection of f onto $\operatorname{span}\{\tilde{P}_0, \tilde{P}_1, \tilde{P}_2\}$. Let φ = best 2^{nd}-degree pol. approximant.

$$\Rightarrow \varphi = \frac{(f, \tilde{P}_0)}{(\tilde{P}_0, \tilde{P}_0)}\tilde{P}_0 + \frac{(f, \tilde{P}_1)}{(\tilde{P}_1, \tilde{P}_1)}\tilde{P}_1 + \frac{(f, \tilde{P}_2)}{(\tilde{P}_2, \tilde{P}_2)}\tilde{P}_2.$$

So $(f, \tilde{P}_\ell)$ can be computed using:

$$\int_0^\pi x \sin x \, dx = \sin x - x \cos x.$$

$$\int_0^\pi x^2 \sin x \, dx = 2x \sin x - (x^2 - 2)\cos x.$$

Relationship with differential operators

Sturm-Liouville operator:

$$\mathcal{L}u = -(pu')' + qu \quad , \qquad p, q \text{ are functions}$$

Linear differential operator, eigenvalues are real $\Rightarrow$ $\mathcal{L}u = \lambda w u$.

Eigenfunctions are orthogonal under the inner product

$(f,g) = \int f(x)\, g(x)\, w(x)\, dx.$ For certain p,q, the eigenfunctions are polynomials.