Goal For a given function \( f \) on \([a, b]\), find \( p_n \in P_n \) that \( \| f - p_n \|_2 \) is as small as possible.

We previously derived that this is a least squares problem and therefore there exists a constructive solution, i.e.,

If \( p_n^{(x)} = c_0 p_0(x) + c_1 p_1(x) + \ldots + c_n p_n(x) \)

form a basis for \( P_n \), then the coefficients \( c_j \) are obtained by solving a linear system.

Finally, if the functions \( q_j \) are orthonormal, then the coefficients \( c_j \) are merely inner products:

\[ c_j = (f, q_j). \quad \text{(analogous to Gram-Schmidt, or orthogonal projections)} \]

This approximation of \( f \) is equivalent to finding its orthogonal projection onto \( P_n \) under the inner product:

\[ (f, g) = \int_a^b f(x) g(x) w(x) \, dx. \]

Definition If the sequence of polynomials \( q_0, q_1, \ldots, q_n \) with \( \deg q_j = j \), on the interval \((a, b)\) satisfies

\[ \int_a^b q_j(x) q_k(x) w(x) \, dx = 0 \quad \text{if} \quad j \neq k, \quad \text{then} \quad q_0, \ldots, q_n \quad \text{is a system of orthogonal polynomials (with weight } \ w(x) = 1 ). \]

Likewise we could define \( (f, g) = \int_a^b f(x) g(x) w(x) \, dx. \)
Example: Find \( q_0, q_1, q_2 \) on \([-1, 1]\) with weight function \( w(x) = 1 \).

Set \( q_0(x) = 1 \).

If \( (q_0, q_1) = 0 \) then
\[
\int_{-1}^{1} z (ax + b) \, dx = 0
\]
\[
2b = 0 \quad \Rightarrow \quad b = 0.
\]

Set \( q_1(x) = x \).

Let \( q_2(x) = x^2 + bx + c \)

Two conditions must be satisfied:
\[
\int_{-1}^{1} q_0(x) q_1(x) \, dx = 0
\]
\[
\int_{-1}^{1} (x^2 + bx + c) \, dx = 0
\]
\[
\frac{2}{3} + 2c = 0
\]
\[
c = \frac{-1}{3}
\]

So we set \( q_2(x) = x^2 - \frac{1}{3} \).

So by construction, \( 1, x, x^2 \) and \( x^2 - \frac{1}{3} \) are orthogonal on \([-1, 1]\).

We could do the same calculation for \( q_3, q_4, \ldots \)

The resulting polynomials are known as Legendre Polynomials. They form an orthogonal basis for all of \( L^2[-1, 1]\) under the inner product \( (f, g) = \int_{-1}^{1} f(x) g(x) \, dx \).

\[
f \in L^2[-1, 1] \iff \int (f(x))^2 \, dx < \infty.
\]

\[
\|f\|_2 < \infty
\]
Legendre polynomials can also be constructed another way as well: the Gram-Schmidt process.

Start with \( P_0(x) = 1 \), \( P_1(x) = x \). \( \int \) automatically orthogonal.

Set \( m_2(x) = x^2 \). \( \exists \) linearly independent from \( P_0, P_1 \).

For Gram-Schmidt: \( P_2 = m_2 - \text{Proj}_\{P_0,P_1\} m_2 \)

\[
= m_2 - \frac{(m_2, P_0)}{(P_0, P_0)} P_0 - \frac{(m_2, P_1)}{(P_1, P_1)} P_1
\]

So \( P_2(x) = x^2 - \frac{2}{3} \cdot \frac{1}{2} \cdot 1 = 0 \)

\[
= x^2 - \frac{1}{3}
\]

Exactly the same as before.

So in general, compute

\[
P_n = m_n - \sum_{k=0}^{n-1} \frac{(m_n, P_k)}{(P_k, P_k)} P_k
\]

\[
P_n(x) = x^n - \sum_{k=0}^{n-1} \frac{P_k(x)}{\|P_k\|_2^2} \int_{-1}^{1} x^n P_k(x) \, dx
\]

These polynomials can be scaled to any interval.

If \( \int_{-1}^{1} P_j(x) P_k(x) \, dx = 0 \) if \( j \neq k \), then it is easy to show that \( \int_{-1}^{1} P_j(t) P_k(t) \, dt = 0 \) if \( j \neq k \)

where \( t = \left( \frac{x+1}{2} \right) \left( b-a \right) + a \).

**Ex:** Chebyshev Polynomials

We know that \( \int_{0}^{\pi} \cos mt \cos nt \, dt = 0 \) if \( m \neq n \).

Let \( t = \cos x \), \( dt = \frac{1}{\sqrt{1-x^2}} \, dx \), \( t : -1 \rightarrow 1 \)
\[
\int_{-1}^{1} \cos(m \cos x) \cdot \cos(n \cos x) \cdot \frac{dx}{\sqrt{1-x^2}} = 0 \quad \text{if } m \neq n.
\]
\[
\Rightarrow \int_{-1}^{1} T_m(x) T_n(x) \frac{1}{\sqrt{1-x^2}} \ dx = 0 \quad \text{if } m \neq n.
\]

Therefore, the functions \( T_0, T_1, T_2, \ldots \) are orthogonal on \([-1,1]\) with respect to the weight function \( w(x) = \frac{1}{\sqrt{1-x^2}} \).

**Theorem** If \( \int_{a}^{b} |f(x)|^2 \, w(x) \, dx < \infty \) (i.e. \( f \in L^2_w(a,b) \)), there is a unique degree \( n \) polynomial \( p_n \) such that

\[
\|f - p_n\|_{w,2} = \min_{q \in \mathbb{P}_n} \|f - q\|_{w,2}
\]

where \( \|f\|_{w,2} = \int_{a}^{b} |f(x)|^2 \, w(x) \, dx \).

**Proof:** Gram-Schmidt solve directly for the coefficients of the associated orthogonal polynomial expansion to compute the approximation.

**Note:** This is just linear algebra!

**Famous sets of orthogonal polynomials:**

<table>
<thead>
<tr>
<th>( w(x) )</th>
<th>( (a,b) )</th>
<th>Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((-1,1))</td>
<td>Legendre</td>
</tr>
<tr>
<td>( \sqrt{1-x^2} )</td>
<td>((-1,1))</td>
<td>Chebyshev</td>
</tr>
<tr>
<td>( e^{-x} )</td>
<td>((0,\infty))</td>
<td>Laguerre</td>
</tr>
<tr>
<td>( e^{-x^2} )</td>
<td>((-\infty,\infty))</td>
<td>Hermite</td>
</tr>
</tbody>
</table>
Compute the best quadratic $L^2$ norm approximation to $f(x) = \sin x$ on $(0,\pi)$ with weight function $w(x) = 1$.

The first 3 Legendre polynomials on $(-1,1)$ are:

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = x^2 - \frac{1}{3}$

To define these functions on $(0,\pi)$, let $t = \frac{\alpha + 1}{2}$

$\Rightarrow \alpha = \frac{2t - 1}{\pi} - 1$

So the shifted orthogonal polynomials are:

- $\tilde{P}_0(t) = 1$
- $\tilde{P}_1(t) = x = \frac{2t - 1}{\pi} - 1 = \frac{2t}{\pi}(t - \frac{\pi}{2})$
- $\tilde{P}_2(t) = x^2 - \frac{1}{3} = \left(\frac{2t}{\pi} - 1\right)^2 - \frac{1}{3}$
  
  $= \frac{4t^2}{\pi^2} - \frac{4t}{\pi} + 1 - \frac{1}{3}$

  $= \frac{4}{\pi}\left(\frac{t^2}{\pi} - t + \frac{\pi}{4}\right)$.

The best approximation is given by the projection of $f$ onto $\text{span}\{\tilde{P}_0, \tilde{P}_1, \tilde{P}_2\}$. Let $q = \text{best 2nd-degree pol. approx.}$

$\Rightarrow q = \frac{\left(f, \tilde{P}_0\right)}{\left(\tilde{P}_0, \tilde{P}_0\right)}\tilde{P}_0 + \frac{\left(f, \tilde{P}_1\right)}{\left(\tilde{P}_1, \tilde{P}_1\right)}\tilde{P}_1 + \frac{\left(f, \tilde{P}_2\right)}{\left(\tilde{P}_2, \tilde{P}_2\right)}\tilde{P}_2$.

So $(f, \tilde{P}_k)$ can be computed using:

- $\int_0^{\pi} x \sin x \, dx = -x \cos x + \sin x$
- $\int_0^{\pi} x^2 \sin x \, dx = 2x \sin x - (x^2 - 2) \cos x$.
Relationship with differential operators

Sturm-Liouville operator:

\[ L u = - (p u')' + q u, \quad p, q \text{ are functions} \]

Linear differential operator, eigenvalues are real \( \Rightarrow Lu = \lambda w u \).

Eigenfunctions are orthogonal under the inner product

\[ (f, g) = \int f(x) g(x) w(x) \, dx. \] For certain \( p, q \), the eigenfunctions are polynomials.