Numerical Analysis

Last time: minimax function approximation

Goal: Find $p_n \in P_n$ such that

$$\|f - p_n\|_{\infty} = \min_{q \in P_n} \max_{x \in [a,b]} |f(x) - q(x)|$$

Thus let $n \geq 0$, $f(x) = x^{n+1}$, then $\|f - p_n\|_{\infty}$ is minimized when $p_n(x) = x^{n+1} - \frac{1}{2^n} \cos((n+1)\cos x)$. 

$$T_{n+1} = \text{Chebyshev polynomial,}$$

Ex: $n = 0 \Rightarrow f(x) = x$

$$p_0(x) = x - \frac{1}{2^0} \cdot \cos \left( a(x) x \right)$$

$$= x - 1 \cdot x = 0$$

Chebyshev polynomials

$$T_n(x) = \cos \left( n a(x) x \right) \quad n = 0, 1, 2, ...$$

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = \cos(2a(x) x)$$

$$T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x)$$

Usually only concerned with $T_n(x)$ for $x \in [-1, 1]$.  $2^n$ polynomial of degree $n+1$ can prove using trig identities applied to this definition.
Trivially, the zeros of $T_n$ can be computed as:

$$\cos(n \cos x) = 0$$

$$\Rightarrow \ n \cos x = \frac{\pi}{2}(2m+1) \quad \text{for } m = -2, -1, 0, 1, 2, \ldots$$

$$\cos x = \frac{\pi}{2n}(2m+1)$$

$$x = \cos\left(\frac{(2m+1)n\pi}{2n}\right) \quad m = 0, 1, \ldots, n-1 \quad \text{(roots repeat for } m \geq n\right).$$

The roots on $[-1, 1]$ can be ordered from $[-1, 1]$ as:

$$x_j = -\cos\left(\frac{(2j-1)n\pi}{2n}\right) \quad j = 1, \ldots, n \quad \text{(n roots)}.$$  

This is an angle, as $j = 1 \ldots n$, we get equispaced points in $[0, \pi]$.

Claim: Interpolation of a function $f$ on $[a, b]$ with a degree $n$ polynomial $p_n$ at the roots of $T_n$ — i.e., Chebyshev nodes — yields a near minimax polynomial approximant.

Idea: The approximation error of the interpolation is

$$f(x) - p_n(x) = \sum_{j=0}^{n} \frac{f^{(m_j)}(z)}{(n+1)!} (x - x_j)$$

If $x_j$ are chosen to be the roots of $T_n$ (properly scaled to this interval), then

$$\sum_{j=0}^{n} (x - x_j) = \frac{d}{dn} T_n(x)$$

Factorization of $\frac{d}{dn} T_n(x)$.  

\[ \text{[2]} \]
What is special about \( \frac{1}{2^n} T_n(x) \)?

It can be shown that \( \frac{1}{2^n} T_n \) is the minimum norm monic polynomial. A monic polynomial of degree \( n \) is one whose coefficient on the \( x^n \) term is 1.

\[ \| f - p_n \|_\infty \leq \frac{M_{n+1}}{(n+1)!} \prod_{j=0}^{n} (x-x_j) \|_{\infty} \]

just minimize this by choosing the "best" \( x_j \)'s.

Approximation in the 2-norm

The 2-norm of a function, with some general continuous weight function \( w > 0 \), on \( (a,b) \), is:

\[ \| f \|_2^2 = \int_a^b (f(x))^2 w(x) \, dx. \]

Goal: Find \( p_n \in P_n \) such that \( \| f - p_n \|_2 \) is minimized. This is a least squares approximation to \( f \) — exactly analogous to solving least squares problems in finite dimensions (i.e., linear algebra).

Therefore the solution is obtained by computing the orthogonal projection of \( f \) onto the space \( P_n \), under the inner product:

\[ (f,g)_w = \text{inner product of } f \text{ with } g \]

\[ = \int_a^b f(x) g(x) w(x) \, dx. \]

There is no reason why the best \( p_n \) for the 2-norm error is the same \( p_n \) for the \( \infty \)-norm error. In more detail...
Our goal: For a function \( f \in L^2(a, b) \) (\( \|f\|_2 = \int_a^b (f(x))^2 w(x) \, dx \leq \infty \)) find \( p_n \in \mathbb{P}_n \) such that:

\[
\|f - p_n\|_2^2 = \inf_{q \in \mathbb{P}_n} \|f - q\|_2^2
= \inf_{q \in \mathbb{P}_n} \int_a^b (f(x) - q(x))^2 w(x) \, dx.
\]

Ex: Let \( n = 0 \), \( f(x) = -2x^2 \) on \([-1, 1]\), \( w(x) = 1 \).
Find \( p_0 = c \) to minimize \( \|f - p_0\|_2^2 \).

\[
\|f - p_0\|_2^2 = \int_{-1}^1 (-2x^2 - c)^2 \, dx
= 2c^2 + \frac{8c}{3} + \frac{8}{5}
\]

Find \( c \) to minimize this quantity.

\[
\frac{d}{dc} \|f - p_0\|_2^2 = 4c + \frac{8}{5} = 0
\Rightarrow c = -\frac{2}{3}
\]

What would the best approximation in \( \| \cdot \|_{\infty} \) be?

Computing the best 2-norm approximation

Recall: The least squares solution to \( Ax = b \) is obtained by solving \( A^TQ^Tb \).

projection of \( b \) onto \( \text{col} \, A \), \( Q \) is obtained by applying the Gram-Schmidt process to the columns of \( A \).

To apply Gram-Schmidt, all that was needed was a vector space and an inner product.
We can do the same thing for polynomial least squares:

1. Trivially, \( P_n \) is a vector space.

2. We can define an inner product on \( P_n \)

\[
(f, g) = \int_a^b f(x) g(x) w(x) \, dx.
\]

Two functions are orthogonal if \( (f, g) = 0 \).

Let \( p_0, p_1, p_2, \ldots, p_n \) be a basis for \( P_n \).

The \( 2 \)-norm approximation problem takes the form:

For \( q(x) = \sum_{j=0}^{n} c_j p_j(x) \), \( \|f - q\|_2^2 \) is given by:

\[
|\text{now} \quad \text{for w} = 1
\]

\[
A = \int_a^b (f(x) - \sum_{j=0}^{n} c_j p_j(x))^2 \, dx
\]

\[
= \int_a^b (f(x))^2 \, dx - 2 \sum_{j=0}^{n} c_j \int_a^b f(x) p_j(x) \, dx + \sum_{j=0}^{n} c_j \sum_{k=0}^{n} c_k \int_a^b p_j(x) p_k(x) \, dx
\]

\[
= (f, f) - 2 \sum_{j=0}^{n} c_j (f, p_j) + \sum_{j=0}^{n} \sum_{k=0}^{n} c_j c_k (p_j, p_k)
\]

Writing down

\[
\nabla A = 0
\]

we have that:

\[
\frac{\partial A}{\partial c_0} = 0, \quad \frac{\partial A}{\partial c_1} = 0, \ldots, \quad \frac{\partial A}{\partial c_n} = 0
\]

\[
\Rightarrow \frac{\partial A}{\partial c_k} = -2 (f, p_k) + 2 \sum_{j=0}^{n} c_j (p_j, p_k) = 0 \quad \text{linear}
\]

\[
\Rightarrow \sum_{k=0}^{n} c_k (p_k, p_k) = (f, p_k) \quad \text{linear equations, in n+1 variables c_0, \ldots, c_n}
\]

In matrix form:

\[
\begin{pmatrix}
(p_0, p_0) & \cdots & (p_0, p_n) \\
(p_1, p_0) & \cdots & (p_1, p_n) \\
\vdots & & \vdots \\
(p_n, p_0) & \cdots & (p_n, p_n)
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_n
\end{pmatrix}
= \begin{pmatrix}
(f, p_0) \\
(f, p_1) \\
\vdots \\
(f, p_n)
\end{pmatrix}
\]
Therefore once this linear system has been solved, the best 2-norm approximation to $f$ is:

$$q = c_0 p_0 + c_1 p_1 + \ldots + c_n p_n.$$  

If the $p_i$'s were orthonormal, i.e. $(p_i, p_j) = \delta_{i,j}$, then the above system is of the form:

$$I \alpha = \text{RHS}.$$  

$S_\alpha = \alpha = (f, p_\alpha).$ (if $p_i$'s are orthonormal).