April 14, 2020

Numerical Analysis

Last time: minimax function approximation

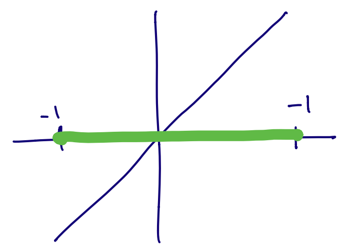
Goal: Find $p_n \in P_n$ such that

$$\|f-p_n\|_\infty = \min_{q \in P_n} \max_{x \in [a,b]} |f(x)-q(x)|$$

Thm Let $n \geq 0$, $f(x) = x^{n+1}$, then $\|f-p_n\|_\infty$ is minimized when $p_n(x) = x^{n+1} - \frac{1}{2^n} \underbrace{\cos((n+1)\operatorname{acos} x)}_{T_{n+1} = \text{Chebyshev polynomial.}}$.

Ex: $n=0$. $\Rightarrow$ $f(x) = x$

$$p_n(x) = x - \frac{1}{2^0} \cdot \cos(\operatorname{acos} x)$$
$$= x - 1 \cdot x = 0$$



Chebyshev polynomials

$T_n(x) = \cos(n \operatorname{acos} x)$ $\quad n = 0, 1, 2, \ldots$

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = \cos(2 \operatorname{acos} x)$

$\vdots$

$\underbrace{T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x)}_{\text{can prove using trig identies applied to this definition}}$ $\leftarrow$ polynomial of degree $n+1$

Usually only concerned with $T_n(x)$ for $x \in [-1,1]$.

Trivially, the zeros of T_n can be computed as:

$$\cos(\underbrace{n \operatorname{acos} x}) = 0$$

$$\Rightarrow \quad n \operatorname{acos} x = \frac{\pi}{2}(2m+1) \qquad \text{for } m = \dots -2, -1, 0, 1, 2 \dots$$

$$\operatorname{acos} x = \frac{\pi}{2n}(2m+1)$$

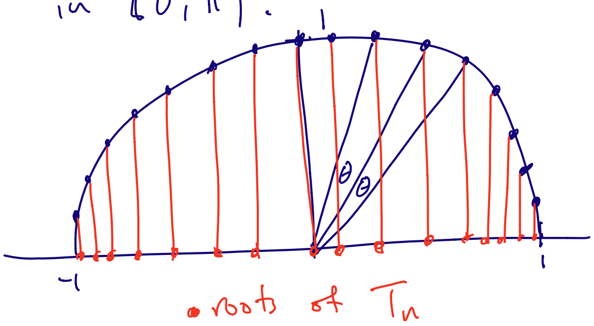
$$x = \cos\left(\frac{(2m+1)(\pi)}{2n}\right) \qquad m = 0, 1, \dots, n-1 \quad \left(\text{roots repeat for } m \geq n\right).$$

The roots on $[-1,1]$ can be ordered from $[-1,1]$ as:

$$x_j = -\cos\left(\underbrace{\frac{(2j-1)\pi}{2n}}\right) \qquad j = 1, \dots, n \qquad (n \text{ roots}).$$

this is an angle, as $j = 1 \dots n$, we get equispaced points in $(0, \pi)$.

$\Rightarrow$



Claim: Interpolation of a function f on $[a,b]$ with a degree n polynomial p_n at the roots of T_n – i.e. Chebyshev nodes – yields a near minimax polynomial approximant.

Idea: The approximation error of the interpolation is

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j)$$

If x_j are chosen to be the roots of T_n (properly scaled to this interval), then $\underbrace{\prod_{j=0}^{n}(x - x_j)} = \frac{1}{2^n} T_n(x)$

Factorization of $\frac{1}{2^n} T_n(x)$.

What is special about $\frac{1}{2^n} T_n(x)$?

It can be shown that $\frac{1}{2^n} T_n$ is the minimum norm monic polynomial. A monic polynomial of degree n is one whose coefficient on the x^n term is 1.

$$\|f - p_n\|_\infty \le \frac{M_{n+1}}{(n+1)!} \underbrace{\left\| \prod_{j=0}^{n} (x - x_j) \right\|_\infty}$$

just minimize this by choosing the "best" x_j's.

Approximation in the 2-norm

The 2-norm of a function, with some general continuous weight function $w > 0$, on (a,b) is:

$$\|f\|_2^2 = \int_a^b (f(x))^2 \, w(x) \, dx.$$

Goal: Find $p_n \in P_n$ such that $\|f - p_n\|_2$ is minimized. This is a least squares approximation to f – exactly analogous to solving least squares problems in finite dimensions (ie. Linear Alg.)

Therefore the solution is obtained by computing the orthogonal projection of f onto the space P_n, under the inner product:

$$(f, g)_w = \text{inner product of } f \text{ with } g$$

$$= \int_a^b f(x) \, g(x) \, w(x) \, dx.$$

Easy to check that this is indeed an inner product

There is no reason why the best p_n for the 2-norm error is the same p_n for the ∞-norm error. In more detail ...

<u>Our goal</u>: For a function $f \in L^2_w(a,b)$ $\left(\|f\|_2 = \sqrt{\int_a^b (f(x))^2 w(x)\, dx} < \infty \right)$

find $p_n \in \mathbb{P}_n$ such that:

$$\|f-p_n\|_2^2 = \inf_{q \in \mathbb{P}_n} \|f-q\|_2^2$$
$$= \inf_{q \in \mathbb{P}_n} \int_a^b \left(f(x) - q(x)\right)^2 w(x)\, dx.$$

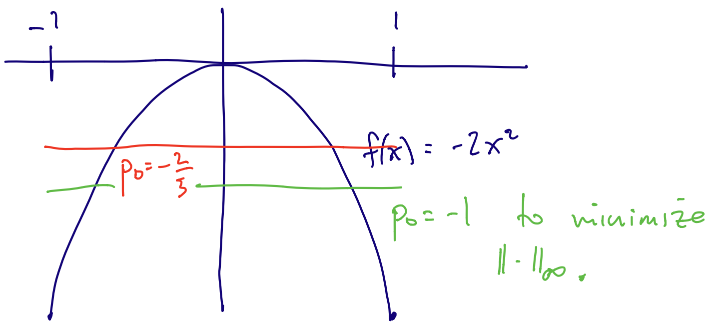
<u>Ex</u>: Let $n=0$, $f(x) = -2x^2$ on $[-1,1]$, $w(x)=1$.

Find $p_0 = c$ to minimize $\|f-p_0\|_2^2$.

$$\|f-p_0\|_2^2 = \int_{-1}^{1} \left(-2x^2 - c\right)^2 dx$$
$$= 2c^2 + \frac{8c}{3} + \frac{8}{5}$$

Find c to minimize this quantity.

$$\frac{d}{dc}\|f-p_0\|_2^2 = 4c + \frac{8}{5} = 0$$
$$\Rightarrow c = -\frac{2}{3}$$



What would the best approximation in $\|\cdot\|_\infty$ be?

<u>Computing the best 2-norm approximation</u>

<u>Recall</u>: The least squares solution to $A\vec{x} = \vec{b}$ is obtained by solving $A\vec{x} = \underbrace{QQ^T\vec{b}}_{\text{projection of } \vec{b} \text{ onto col } A}$, Q is obtained by applying the Gram-Schmidt process to the columns of A.

To apply Gram-Schmidt, all that was needed was a vector space and an inner product.

We can do the same thing for polynomial least squares:

① Trivially, P_n is a vector space.

② We can define an inner product on P_n by: $(f,g) = \int_a^b f(x)\, g(x)\, w(x)\, dx$.

Two functions are orthogonal if $(f,g)=0$.

Let $p_0, p_1, p_2, \dots, p_n$ be a basis for P_n.

The 2-norm approximation problem takes the form:

For $q(x) = \sum_{j=0}^{n} c_j p_j(x)$, $\|f-q\|_2^2$ is given by: (set $w=1$ for now)

$$A = \int_a^b \left(f(x) - \sum_{j=0}^{n} c_j p_j(x)\right)^2 dx$$

$$= \int_a^b (f(x))^2\, dx - 2\sum_{j=0}^{n} c_j \int_a^b f(x)\, p_j(x)\, dx + \sum_{j=0}^{n}\sum_{k=0}^{n} c_j c_k \int_a^b p_j(x)\, p_k(x)\, dx$$

$$= (f,f) - 2\sum_{j=0}^{n} c_j (f,p_j) + \sum_{j=0}^{n}\sum_{k=0}^{n} c_j c_k (p_j,p_k)$$

Writing down

$\nabla A = \vec{0}$, we have that:

$$\frac{\partial A}{\partial c_0} = 0, \quad \frac{\partial A}{\partial c_1} = 0, \quad \dots, \quad \frac{\partial A}{\partial c_n} = 0$$

$$\Rightarrow \frac{\partial A}{\partial c_\ell} = -2(f,p_\ell) + 2\sum_{k=0}^{n} c_k (p_\ell,p_k) = 0$$

$$\Rightarrow \sum_{k=0}^{n} c_k (p_\ell, p_k) = (f, p_\ell)$$

$\leftarrow$ $n+1$ linear equations, in $n+1$ variables $c_0, \dots, c_n$.

In matrix form:

$$\begin{pmatrix} (p_0,p_0) & \cdots & (p_0,p_n) \\ (p_1,p_0) & \cdots & (p_1,p_n) \\ \vdots & & \\ (p_n,p_0) & \cdots & (p_n,p_n) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} (f,p_0) \\ (f,p_1) \\ \vdots \\ (f,p_n) \end{pmatrix}$$

Therefore once this linear system has been solved, the best 2-norm approximation to f is:

$$q = c_0 p_0 + c_1 p_1 + \dots + c_n p_n.$$

<u>If</u> the p_ℓ's were orthonormal, i.e. $(p_\ell, p_k) = \delta_{\ell k}$, then the above system is of the form:

$$I \vec{c} = \text{RHS}.$$

So $= c_\ell = (f, p_\ell)$. (if p_ℓ's are orthonormal).