That was for vectors – how about for matrices?

Def: Matrix norm $\|A\|$: (same rules !)

(1) $\|A\| \geq 0$

(2) $\|\alpha A\| = |\alpha| \|A\|$

(3) $\|A+B\| \leq \|A\| + \|B\|$

[(4) $\|AB\| \leq \|A\| \cdot \|B\|$ (sometimes)]

If $\|\underline{u}\|$ is any norm on a vector, then the induced matrix norm is:

$$\|A\| = \max_{\|\underline{u}\|=1} \|A\underline{u}\| = \max \frac{\|A\underline{u}\|}{\|\underline{u}\|}$$

$\Rightarrow$ For any $\|\underline{u}\|$, $\|A\underline{u}\| \leq \|A\| \, \|\underline{u}\|$.

Thm: If $\|\underline{u}\| = \|\underline{u}\|_1 = \sum |u_i|$ then the induced norm ~~on A~~ on matrix is:

$$\|A\| = \max_{j=1,n} \sum_{i=1}^{m} |a_{ij}| \qquad \left(\begin{array}{l}\text{max over columns} \\ \text{sum over rows}\end{array}\right)$$

Pf: $\|A\underline{u}\| = \|\sum \underline{a}_j u_j\| \leq \sum \|\underline{a}_j\| \, |u_j| \leq \max \|\underline{a}_j\| \sum |u_j|$

$= \max \|\underline{a}_j\| \, \|\underline{u}\|$

But $\underline{u} = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \Rightarrow \max \|\overset{A\underline{u}}{\underline{a}_j}\| \geq \|a_1\|$ $\square$

Alternatively:

Thm $\|A\|_\infty = \max_{i=1,m} \sum_{j=1}^{n} |a_{ij}|$

Pf: $\|A\underline{u}\|_\infty = \max_{i=1,m} |(A\underline{u})_i| = \max_i |\sum_j a_{ij} u_j|$

$$\le \max_i \sum_j |a_{ij}| \, |u_j|$$

$$\le \max_i \left(\max_j u_j\right) \sum_j |a_{ij}|$$

$$= \|\underline{u}\|_\infty \max_i \sum_j |a_{ij}|$$

$\Leftarrow$ set $\underline{u} = \underline{1}$ to pick out maximal row.

Lastly, and most commonly used:

Thm $\|A\|_2 = \sqrt{\max_j \lambda_j}$ of A^tA

Pf $\|A\underline{u}\|_2^2 = (A\underline{u}, A\underline{u}) = (\underline{u}, A^tA\underline{u})$

A^tA is SPD, and has <u>real</u> eigenvalues

$\Rightarrow$ diagonalizable $\Rightarrow (\underline{u}, PDP^t\underline{u})$

$$= (P^t\underline{u}, DP^t\underline{u})$$

$\max_{\underline{u}} (P^t\underline{u}, DP^t\underline{u}) = \max_{\underline{v}=1} (\underline{v}, D\underline{v}) =$ largest eigenvalue.

Condition number of a problem

$\Rightarrow$ The sensitivity of the "problem" ~~to~~ at the solution.

Ex: For a function $y = f(x)$, how sensitive is y to x? ~~→ error estimate is the rate~~ (The "problem" is computing $f(x)$.)

① In an absolute sense:

$$|y - y'| \approx \underbrace{C(x)}_{\text{absolut condition number.}} |x - x'|$$

$$\Rightarrow C(x) \approx \frac{|y - y'|}{|x - x'|} \cancel{\approx \ldots}$$

$$= \frac{|f(x) - f(x')|}{|x - x'|}$$

for x' close to x, $C(x) \approx |f'(x)|$.

② In a relative sense:

$$\frac{|y' - y|}{|y|} \approx K(x) \left| \frac{x' - x}{x} \right|$$

$\frac{|x - x'|}{|x|}$ = relative error

$$\Rightarrow K(x) \approx \left|\frac{y' - y}{y}\right| \left|\frac{x}{x' - x}\right|$$

$$= \left|\frac{f(x') - f(x)}{x' - x}\right| \left|\frac{x}{y}\right| = \left|\frac{x f'(x)}{f(x)}\right|$$

Ex: Let $y = x^{\frac{1}{3}} = f'(x)$

$$C(x) = \frac{1}{3} x^{-2/3}$$

$$K(x) = \frac{x f'(x)}{f(x)} = \frac{x \cdot \frac{1}{3} \cdot x^{-2/3}}{x^{1/3}} = \frac{\frac{1}{3} x^{1/3}}{x^{1/3}} = \frac{1}{3}$$

$C(x) = \frac{1}{3} \frac{1}{x^{2/3}} < \infty$ for ~~x~~ x away from 0

$K(x)$ is small everywhere

$\cancel{f(x)}$ $+\epsilon$

$$f(x+\epsilon) \approx f(x) + f'(x)\epsilon$$

$$\approx f(x) + \frac{1}{3} x^{-2/3} \epsilon$$

$$\Rightarrow |f(x+\epsilon) - f(x)| \approx \underbrace{\frac{1}{3x^{\frac{2}{3}}}}_{\text{large for } x \text{ near } 0.}\epsilon$$