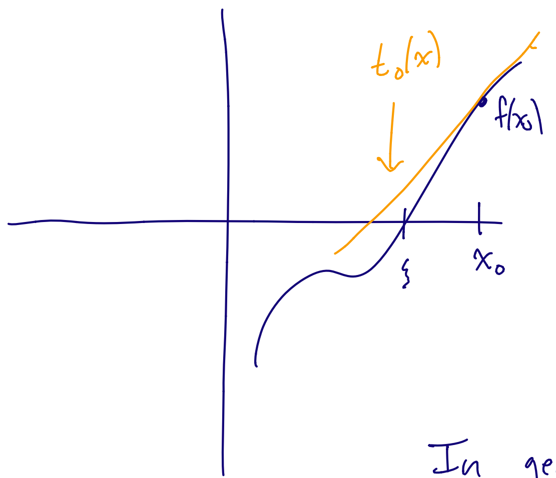
Feb 4, 2020 Numerical Analysis

Today: Analysis of Newton's Method

Recall: Newton's Method approximates a function by its tangent line and then finds the root of the tangent line. And then repeats this:



$f(\xi)=0$

$t_0(x) = f(x_0) + f'(x_0)(x-x_0)$

$$t_0(x_1)=0 \implies \underbrace{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}_{\text{Newton iteration}}$$

In general, Newton's method is:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Two questions to ask:

(1) Does Newton's method converge?

I.e. Does the sequence $\{x_k\}$ converge to ξ?

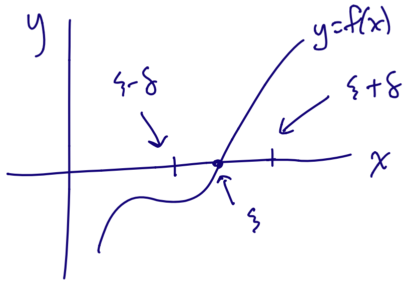
I.e. Is $\lim_{k\to\infty} x_k = \xi$?

(2) If it converges, how fast does it converge.

We answer these questions via Theorem / Proof, placing certain assumptions on f.

Theorem (1.8 from Suli & Mayers)

Suppose that f is twice continuously differentiable (i.e. f, f', and f'' are continuous) on the interval $I_\delta = [\xi-\delta, \xi+\delta]$, $\delta > 0$, and that $f(\xi)=0$ and $f''(\xi) \neq 0$.



Also assume that there exists $A > 0$ such that $\left|\frac{f''(x)}{f'(y)}\right| \leq A$ for all $x, y \in I_\delta$.

If $|\xi - x_0| \leq h$, with $h \leq \min(\delta, \frac{1}{A})$, then the sequence $\{x_k\}$ defined by Newton's Method $x_{k+1} = x_k - f(x_k)/f'(x_k)$, (with starting guess x_0) converges quadratically to ξ.

Proof By Taylor's Theorem,

$\eta_k \in [\xi, x_0] \subseteq I_\delta$

$$f(\xi) = 0 = f(x_0) + f'(x_0)(\xi - x_0) + \frac{f''(\eta_0)}{2}(\xi - x_k)^2.$$

And since by Newton's Method $\xi - x_1 = \xi - x_0 + \frac{f(x_0)}{f'(x_0)}$,

$$(*) \quad \Rightarrow \quad \xi - x_1 = -\frac{f''(\eta_0)}{2f'(x_0)}(\xi - x_0)^2$$

And therefore
$$|\xi - x_1| \leq \frac{1}{2}\left|\frac{f''(\eta_0)}{f'(x_0)}\right| |\xi - x_0|^2$$
$$\leq \frac{1}{2} A \, |\xi - x_0| \cdot |\xi - x_0|$$
$$\leq \frac{1}{2} A \, \frac{1}{A} \, h = \frac{1}{2} h.$$

And once again, $|\xi - x_1| \leq h$, so $\Rightarrow |\xi - x_2| \leq \frac{1}{2^2} h$.

Repeating k times we have that

$$|\xi - x_k| \leq \frac{1}{2^k} h \implies \lim_{k\to\infty} x_k = \xi$$

$$\implies \boxed{\text{convergence}}$$

Furthermore,

since $$|\xi - x_{k+1}| = \frac{1}{2}\left|\frac{f''(\eta_k)}{f'(x_k)}\right| |\xi - x_k|^2$$

we have that

$$\lim_{k\to\infty} \frac{|\xi - x_{k+1}|}{|\xi - x_k|^2} = \lim_{k\to\infty} \frac{1}{2}\left|\frac{f''(\eta_k)}{f'(x_k)}\right| = \frac{1}{2}\left|\frac{f''(\xi)}{f'(\xi)}\right|$$

since $\lim_{k\to\infty} \eta_k = \xi$, since $\eta_k \in [\xi, x_k]$ in Taylor's Thm.

This proves Quadratic Convergence. $\square$

Failures of Newton's Method:

① When Newton's Method converges, and fails to converge, it is usually because $f' = 0$ at the root (or possibly higher derivatives as well).

In this case, the quantity $\frac{f''(x)}{f'(y)}$ may not remain bounded in I_δ

② For some initial guesses, Newton's Method may fail to converge at all.