The Bisection Method used two pieces of information to approximate the solution to \( f(x) = 0 \) — the sign of the function.

What if we use values? The result is the Secant Method.

Start with two guesses for the root, \( x_0, x_1 \),

Graphically:

Find the root of the secant line:

\[
S(x) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0) + f(x_0)
\]

The root of the secant line satisfies \( S(x) = 0 \):

\[
s(x) = 0 \implies x = x_1 - \frac{f(x)}{f(x_1) - f(x_0)} (x_1 - x_0)
\]

Call this root \( x_2 \), the next approximation to the root of \( f \). Also, define \( f(x_2) = f_k \).

The secant method generates a sequence of approximations to the root of \( f \) by:

\[
x_{k+1} = x_k - f_k \left( \frac{x_k - x_{k-1}}{f_k - f_{k-1}} \right)
\]

We will revisit the convergence properties of this method later.

Summary: Approximate \( f \) by a secant line, find root of secant line, repeat using new approximate root.
Newton's Method

What if we are now allowed to use derivative information to approximate \( f \)? (And then find the root of this approximant.)

Suppose we know \( f(x_0) \) and \( f'(x_0) \), then we can draw the tangent line.

\[
f(x) = \text{tangent line}
\]

The equation of the tangent line is given as:

\[
t(x) = f(x_0) + f'(x_0)(x-x_0)
\]

The root of the tangent line satisfies \( t(x) = 0 \)

\[\Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)} = x_1\]

\(x_1\) is the next approximation to the root of \( f \).

Repeating this procedure at \( x_1 \), we obtain Newton's Method:

\[x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}\]

Alternative interpretation: the tangent line is a first-order Taylor approximation to \( f \).

Recall: The Taylor series of \( f \) about \( x_0 \) (assuming \( f \) has infinite derivatives):

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n
\]

\[= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!} (x-x_0)^2 + \frac{f'''(x_0)}{3!} (x-x_0)^3 + \ldots \]
Truncating this series after the first two terms gives:
\[ f(x) \approx f(x_0) + f'(x_0)(x-x_0) \]

If \( x_0 \) is close to the root of \( f \), i.e., \( f(\xi) = 0 \), then we get that
\[ f(\xi) = 0 \approx f(x_0) + f'(x_0)(\xi-x_0) \]
Solve for \( \xi \approx x_0 - \frac{f(x_0)}{f'(x_0)} \). This is Newton's Method.

The Secant Method and Newton's Method both work by the same mechanism: approximate \( f \) by a linear function, find the root of that linear function, update and repeat.

**Convergence Behaviour**

Let \( \xi \) be the true root of \( f \), i.e., \( f(\xi) = 0 \).

We are interested in how the absolute error of \( x_k \) changes from iteration to iteration, \( e_k = |\xi - x_k| \).

For the bisection method:
\[ e_{k+1} \approx \frac{1}{2} e_k \]

It turns out that Newton converges quadratically:
\[ e_{k+1} \approx A e_k^2 \]

If \( e_0 = 10^{-1} \), \( e_3 \approx 10^{-9} \)
\( e_1 \approx 10^{-2} \), \( e_4 \approx 10^{-16} \)
\( e_2 \approx 10^{-4} \), \( e_5 \approx 10^{-16} \)

We will prove this next class.