These notes give a very brief summary of magnetostatics, a topic we will often discuss in this course. All the derivations are inspired from J.P. Freidberg's notes on Magnetoostatics for the NSE class *22.105 Electromagnetic Interactions* at MIT.

Magnetostatics is the field of electrodynamics that describes steady-state magnetic fields that arise from the motion of charged particles. Note that the presence of moving charged particles does not mean that we must also have an electric field. Negative charges can flow through positive charges in such a way that current flows but there is no net charge, hence no electric field.

1 Biot and Savart law

1.1 Current density

In steady-state, the source of magnetic fields are *electric currents*. Electric currents correspond to charges in motion, and are described by the *current density* $\mathbf{J}$ measured in units of positive charge crossing unit area per unit time. For a charge distribution ρ (same notation as in the notes for electrostatics) going at the velocity $\mathbf{v}$, the current density is given by

$$\mathbf{J} = \rho \mathbf{v} \tag{1}$$

1.2 Biot and Savart law

Our starting point, the equivalent of Coulomb's law for electrostatics, is the empirically verified law relating a magnetic field $\mathbf{B}$ and a current density $\mathbf{J}$. This is called the Biot-Savart law and can be written in the following integral form :

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d\mathbf{r}' \tag{2}$$

The multiplicative constant $\mu_0/4\pi$ is linked to our choice to work in SI units. $\mu_0 = 4\pi \times 10^{-7}$henry per meter is called the *magnetic permeability* of free space. In magnetostatics, $\mathbf{B}$ and $\mathbf{J}$ play the role which $\mathbf{E}$ and ρ play in electrostatics.

1.3 The vector potential

Recall from the notes on electrostatics that

$$\nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = - \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

Using this fact, we can rewrite the Biot-Savart law as

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d\mathbf{r}' \tag{3}$$

Now, for any scalar g and any vector $\mathbf{U}$, we have the property $\nabla \times (g\mathbf{U}) = \nabla g \times \mathbf{U} + g \nabla \times \mathbf{U}$. Using this in Equation (3), we have

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) d\mathbf{r}' \tag{4}$$

and since ∇ operates on unprimed quantities, we can take it out of the integral :

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \tag{5}$$

Equation (4) allows us to define the *vector potential* $\mathbf{A}$ as

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' = \mu_0 \int \mathbf{J}(\mathbf{r}') G_L(\mathbf{r}, \mathbf{r}') d\mathbf{r}' \tag{6}$$

where G_L is the free space Green's function for Laplace's equation we introduced in the lecture notes on electrostatics, so that the magnetic field $\mathbf{B}$ can be written as

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{7}$$

Equation (7) obviously implies that

$$\nabla \cdot \mathbf{B} = 0 \tag{8}$$

There are no *free magnetic charges*, or, in other words, no magnetic monopoles.

2 Charge conservation

In this section, we briefly consider time variations (and thereby depart from the "statics" part of electrostatics and magnetostatics) to derive an important relation between ρ and $\mathbf{J}$, which will be useful in the next section to obtain some of the equations of magnetostatics in differential form.

The equation linking ρ and $\mathbf{J}$ comes from the physical principle of charge conservation, which says that electric charge can neither be created nor destroyed. The net quantity of electric charge, the amount of positive charge minus the amount of negative charge in the universe, is always conserved. Having the principle of charge conservation in mind, consider an Eulerian volume V and consider the total charge Q in that volume :

$$Q = \int_V \rho(\mathbf{r}')d\mathbf{r}'$$

Since charge can neither be destroyed nor created, the change of Q in time can only come from the flux of charge in and out of the volume V through the surface ∂V of V. This flux is $\rho\mathbf{v}$, i.e. it is $\mathbf{J}$. We can therefore write :

$$\frac{dQ}{dt} = \int_V \frac{\partial\rho(\mathbf{r}')}{\partial t}d\mathbf{r}' = -\int_{\partial V} \mathbf{J}\cdot\mathbf{n}dS \tag{9}$$

where $\mathbf{n}$ is the unit vector normal to the surface ∂V and the minus sign comes from the fact that the unit normal is pointing outward. Using the divergence theorem in the last term, we have

$$-\int_{\partial V} \mathbf{J}\cdot\mathbf{n}dS = -\int_V \nabla\cdot\mathbf{J}d\mathbf{r}'$$

and Equation (9) can be written as

$$\int_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J}\right)d\mathbf{r}' = 0 \tag{10}$$

Since Equation (10) is true for any Eulerian volume V, we conclude

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 \tag{11}$$

This is the equation of conservation of charge. It is quite intuitive in that it is the equivalent of the equation of conservation of mass (also called continuity equation) in fluid dynamics.

Now, if $\mathbf{J} = 0$, we have $\frac{\partial\rho}{\partial t} = 0$, as expected : this is the electrostatics limit. Conversely, in the magnetostatics limit we have $\frac{\partial\rho}{\partial t} = 0$ which implies

$$\nabla\cdot\mathbf{J} = 0 \tag{12}$$

3 Differential form for the equations of magnetostatics

3.1 Equation for the vector potential

We found

$$\mathbf{A} = \mu_0\int \mathbf{J}(\mathbf{r}')G_L(\mathbf{r},\mathbf{r}')d\mathbf{r}'$$

Taking the divergence of this equation and taking the ∇ inside the integral sign, we have

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \mu_0 \int \nabla \cdot [\mathbf{J}(\mathbf{r}')G_L(\mathbf{r},\mathbf{r}')]\, d\mathbf{r}' \\ &= \mu_0 \int \Big[-\nabla' \cdot [G_L(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')] + G_L(\mathbf{r},\mathbf{r}')\nabla' \cdot \mathbf{J}(\mathbf{r}')\Big] d\mathbf{r}'\end{aligned}$$

where ∇' represents the gradient operator with respect to the primed coordinates. Since $\nabla \cdot \mathbf{J} = 0$, this expression simplifies as

$$\nabla \cdot \mathbf{A} = -\mu_0 \int \nabla' \cdot [\mathbf{J}(\mathbf{r}')G_L(\mathbf{r},\mathbf{r}')]\, d\mathbf{r}' \tag{13}$$

We can use the divergence theorem in Equation (13) to obtain

$$\nabla \cdot \mathbf{A} = -\mu_0 \int_{\partial\Omega} G_L(\mathbf{r},\mathbf{r}')\mathbf{J} \cdot \mathbf{n}(\mathbf{r}')dS = 0 \tag{14}$$

which holds for any *localized* current distribution since Ω is the whole configuration space. We conclude that

$$\nabla \cdot \mathbf{A} = 0 \tag{15}$$

3.2 Analog of Poisson's equation

The analog of Poisson's equation for the vector potential is obtained by taking the vector Laplacian of $\mathbf{A}$:

$$\begin{aligned}\mathbf{A} &= \mu_0 \int \mathbf{J}(\mathbf{r}')G_L(\mathbf{r},\mathbf{r}')d\mathbf{r}' \\ \Rightarrow \nabla^2\mathbf{A} &= \mu_0 \int \mathbf{J}(\mathbf{r}')\nabla^2 G_L(\mathbf{r},\mathbf{r}')d\mathbf{r}' \\ &= -\mu_0 \int \mathbf{J}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')d\mathbf{r}'\end{aligned}$$

so that finally we find

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J} \tag{16}$$

3.3 Equation for the magnetic field

We have the following vector identity :

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

Using Equations (15) and (16) we then immediately find

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J} \tag{17}$$

4 Summary of magnetostatics

The goal in electrostatics is to determine the vector potential $\mathbf{A}$ or equivalently the magnetic field $\mathbf{B}$ due to fixed electric current distributions that do not vary in time. The relationship between $\mathbf{B}$ and $\mathbf{A}$ is $\mathbf{B} = \nabla \times \mathbf{A}$, and $\mathbf{B}$ and $\mathbf{A}$ are determined in terms of the current distribution $\mathbf{J}$ according to the following formulas

— **Integral formulation**

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d\mathbf{r}'$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

— **Differential formulation**

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J}$$

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$$