Homework 3

Due: 12:30pm March 8, 2018

Notes on the assignment:

Submission: Homework assignments must be submitted **via email** to the following address:

`hw3.ymh6j7vzifp8j8it@u.box.com`

by the start of class on the due date. Homework submitted after this time will **not be accepted**. Your submission should take the form of a *single file* with the filename `yourNetID_hw3.zip` attached to an email to the above address. The subject line and body of the email will be ignored, so please do not include any other comments aside from the attached file. For example, if your NYU netID is `abc123`, then you should submit a single file with filename `abc123_hw3.zip`. The `.zip` archive should include a PDF of the written portion of your homework, and any files required for the programming aspect of your homework. Please prepare cleanly handwritten or typed (preferably with LaTeX) homework, and make sure that your name is on the homework. Feel free to use original homework LaTeX document to write-up your homework. If you are required to hand in code, this will explicitly be stated on the homework assignment.

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1. (10 pts) In class, we derived the fact that the 2-norm of a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ can be computed based on an eigenvalue calculation:

$$\|\boldsymbol{A}\|_2 = \max_{\|\boldsymbol{u}\|_2=1} \|\boldsymbol{A}\boldsymbol{u}\|_2 = \max_j \sqrt{\lambda_j}, \tag{1}$$

where λ_j denote the eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$. Prove the final step required in the calculation we did in class, in particular that:

$$\max_{\|\boldsymbol{v}\|_2=1} \sum_{j=1}^{n} \lambda_j v_j^2 = \max_j \lambda_j, \tag{2}$$

where it is assumed that $\lambda_j \geq 0$, and v_j are the components of $\boldsymbol{v}$, i.e. $\boldsymbol{v} = (v_1 \cdots v_n)^T$.

2. Another commonly used matrix norm in linear algebra is the *Frobenius norm*, defined for a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ as:

$$\|\boldsymbol{A}\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}. \tag{3}$$

   (a) (4 pts) Prove that $\|\boldsymbol{A}\|_F = \sqrt{\operatorname{trace}(\boldsymbol{A}^T\boldsymbol{A})}$.

   (b) (3 pts) Prove also that $\|\boldsymbol{A}\|_F = \sqrt{\sum_j \sigma_j^2}$, where σ_j are the singular values of $\boldsymbol{A}$.

   (c) (3 pts) Show that $\|\boldsymbol{A}\|_2 \leq \|\boldsymbol{A}\|_F$.

3. (10 pts) Find the relative condition number (by hand) of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 2 \\ 0 & -3 & 0 \\ 1 & 0 & -1 \end{pmatrix}. \tag{4}$$

4. (10 pts) Let $\boldsymbol{A}$ be an $n \times n$ real-valued symmetric matrix with n distinct non-zero eigenvalues. Prove that $\boldsymbol{A}$ has n linearly independent, orthogonal eigenvectors and that all the eigenvalues are real (i.e. not complex).