Next topic: Stiff equations

Example:

$$y_1' = -100\, y_1 + y_2$$

$$y_2' = \frac{-y_2}{10}$$

$$\Rightarrow \quad \vec{y}\,' = A\vec{y} \quad \Bigg| \quad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -100 & 1 \\ 0 & -1/10 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\Rightarrow \quad y_2(t) = y_2(0)\, e^{-t/10}$$

$$\Rightarrow \quad y_1(t) = \left(y_1(0) - \frac{10}{999} y_2(0)\right) e^{-100t} + \frac{10}{999} y_2(0)\, e^{-t/10}$$

↑ decays extremely fast

Question: how small should the time-step h be?

Try Euler's Method

2^{nd} Eqn: $$y_{2,k+1} = y_{2,k} - \frac{h}{10} y_{2,k}$$

$$\Rightarrow \quad y_{2,k} = \left(1 - \frac{h}{10}\right)^k y_2(0)$$

1st Eqn: $y_{1,k+1} = y_{1,k} + h \cdot (-100\, y_{1,k} + y_{2,k})$

$$= (1 - 100h)\, y_{1,k} + h \cdot y_{2,k}$$

$$= (1 - 100h)\, y_{1,k} + h \cdot \left(1 - \frac{h}{10}\right)^k y_2(0)$$

Continue back substitution for $y_{1,k}$ to obtain

$$y_{1,k+1} = (1-100h)^{k+1} y_1(0) + h\left(1 - \frac{h}{10}\right)^k \left(\sum_{l=0}^{k} \left(\frac{1-100h}{1-h/10} \right)^l \right) y_2(0)$$

$$\approx c_1 (1-100h)^{k+1} + c_2 \left(1 - \frac{h}{10}\right)^{k+1}$$

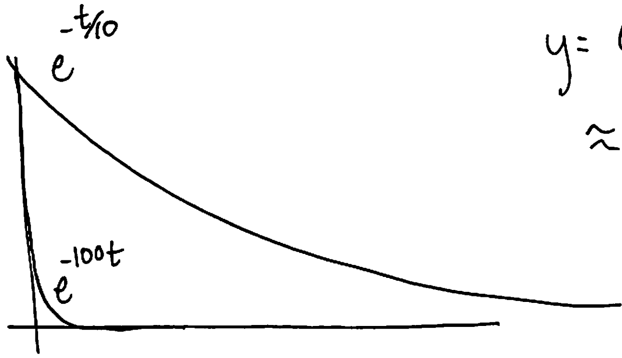
We need **both** $|1-100h| < 1$ and $\left|1 - \frac{h}{10}\right| < 1$

($|1-100h| < 1$) determines maximum time-step

$$\Rightarrow h < \frac{1}{50}$$

If $h > \frac{1}{50}$, the term $(1-100h)^{k+1}$ will grow **even after** e^{-100t} has decayed to near zero.

This is the defining characteristic of stiff equations: the solution has components which behave ~~only~~ on vastly different time scales.



$$y = e^{-t/10} + e^{-100t}$$

$\approx e^{-t/10}$ for large t, but still requires time-steps dictated by $e^{-100t} \approx 0$.

More Stability Analysis

Consider the model problem:

$$y' = \lambda y$$

The solution is $y(t) = ce^{\lambda t} \to 0$ iff $Re(\lambda) < 0$.

This is a very simple example, but similar to the problem

$$\vec{y}\,' = A\vec{y}$$

when A can be diagonalized: $A = PDP^{-1}$

In this case,

$$\vec{y}' = PDP^{-1}\vec{y}$$

$$\underbrace{(P^{-1}\vec{y})'}_{\vec{w}} = D\underbrace{(P^{-1}\vec{y})}_{\vec{w}}$$

$$\Rightarrow \quad \vec{w}' = D\vec{w} \quad \Rightarrow \quad \begin{aligned} w_1' &= \lambda_1 w_1 \\ w_2' &= \lambda_2 w_2 \\ &\vdots \\ w_n' &= \lambda_n w_n \end{aligned}$$

Often, the matrix A will be the <u>linear approximation</u> to the true non-linear initial value problem.

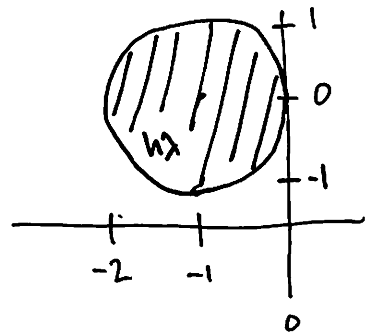
<u>Definition</u> ~~[illegible]~~

<u>Region of absolute stability</u>: All numbers $z = h\lambda \in \mathbb{C}$ such that $y_k \to 0$ as $k \to \infty$ when the ODE method is applied to $y' = \lambda y$.

Example: Euler's Method

$$y_{k+1} = y_k + h\lambda y_k$$
$$= (1+h\lambda) y_k$$
$$= (1+h\lambda)^k y_0$$

$|1+h\lambda| < 1 \iff$



Ex 2: If the eigenvalues of A are

$\lambda_1 = -1 + 10i$

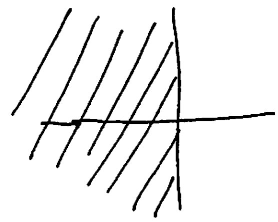
$\lambda_2 = -10 - 10i$ then h must satisfy

$$|1 + h(-1+10i)| < 1$$
$$|1 + h(-10-10i)| < 1$$

$\iff (1-h)^2 + 100h^2 < 1 \implies h < 2/101$ ← this condition controls h.

$(1-10h)^2 + 100h^2 < 1 \implies h < 1/10$

If the region of stability is the left-half-plane



then for any λ with $Re(\lambda) < 0$, for any $h > 0$ the solution decays $\Rightarrow$ A-stable

Example Backward Euler:

$$y_{k+1} = y_k + h\lambda y_{k+1}$$

solve for y_{k+1}: $\quad y_{k+1} = \frac{1}{1-h\lambda} y_k \cdots = \frac{1}{(1-h\lambda)^{k+1}} y_0$

$\Rightarrow$ region of stability is

$$\frac{1}{|1-h\lambda|} < 1 \iff |1-h\lambda| > 1$$

If $Re(\lambda) < 0$, then $|1-h\lambda| > 1$ for any $h > 0$.

Lastly There are no explicit A-stable methods of the form $\sum a_\ell y_{k+\ell} = h \sum b_\ell f(t_{k+\ell}, y_{k+\ell})$

Compromise: Find high-order method with a large region of stabili