Lecture 24 Numerical Analysis April 26, 2018

Another brute-force method for computing derivatives:

Richardson extrapolation: (Not just for ~~the~~ differentiation, can be used for any method with some $O(h^n)$ accuracy)

Idea: What if we compute $f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$ for several values of h — can we use this information to get a better estimate of f'?

Recall: $$\underbrace{\frac{f(x+h) - f(x-h)}{2h}}_{} = f'(x) + \frac{h^2}{6} f^{(3)}(x) + O(h^4)$$

call this $\varphi_0(h)$.

$$\varphi_0\left(\frac{h}{2}\right) = \frac{f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)}{2\left(\frac{h}{2}\right)} = f'(x) + \frac{1}{4}\frac{h^2}{6} f^{(3)}(x) + O(h^4)$$

Can we take a linear combination of $\varphi_0(h)$ and $\varphi_0(\frac{h}{2})$ to kill the $O(h^2)$ term?

$$4 \cdot \varphi_0(\tfrac{h}{2}) - \varphi_0(h) = 3 f'(x) + O(h^4)$$

$$\Rightarrow \quad f'(x) = \frac{4\varphi_0(\frac{h}{2}) - \varphi_0(h)}{3} + O(h^4)$$

$\Rightarrow$ min error on the order of $O(\epsilon^{4/5}) \approx 10^{-13}$

This can be repeated!

$$\varphi_1(\tfrac{h}{2}) \quad \frac{4 \cdot \varphi_0(\frac{h}{4}) - \varphi_0(\frac{h}{2})}{3} = f'(x) + O\left(\frac{h^4}{16}\right)$$

$$\varphi_1(h) \quad \frac{4 \cdot \varphi_0(\frac{h}{2}) - \varphi_0(h)}{3} = f'(x) + O(h^4)$$

$$\Rightarrow \quad \frac{16\,\varphi_1(\frac{h}{2}) - \varphi_1(h)}{15} = f'(x) + O(h^6)$$

General use of Richardson extrapolation:

If some quantity L has truncation error (or order of convergence):

$$L = \varphi_0(h) + a_1 h + a_2(h^2) + a_3 h^3 + \ldots$$

then $$L = \varphi_0\left(\frac{h}{2}\right) + \frac{a_1}{2} h + \frac{a_2}{4} h^2 + \frac{a_3}{8} h^3 + \ldots$$

$$\Rightarrow 2\varphi_0\left(\frac{h}{2}\right) - \varphi(h) \simeq L + O(h^2)$$

$$\vdots$$

This analysis does <u>not</u> include round off effects.

Very powerful, brute-force method.

<u>Questions</u>?

Error analysis: Local vs. Global

Local truncation error:

Euler's method

$$\frac{y_{k+1} - y_k}{h} = f(t_k, y_k)$$

For local error, write difference eq'n in this form

True ODE

$$\frac{y(t_{k+1}) - y(t_k)}{h} = f(t_k, y(t_k)) + \frac{h}{2} y''(\xi_k)$$

local error

First order

Euler's method is consistent: $\lim_{h \to 0} \frac{y(t_{k+1}) - y(t_k)}{h} = y'(t_k) = f(t_k, y(t_k)).$

Global error: $\max_{\substack{k \\ t \in [t_0, T]}} |y(t_k) - y_k|$ (max error on the entire interval you are solving)

Euler: $O(h)$ ~~proof is in the book,~~

Proof:

Proof of Global Error for Euler:

Euler: $y_{k+1} = y_k + h f(t_k, y_k)$

Taylor: $y(t_{k+1}) = y(t_k) + h f(t_k, y(t_k)) + \frac{h^2}{2} y''(\xi_k)$

Taylor − Euler:
$$d_{k+1} = y(t_{k+1}) - y_k$$
$$= d_k + h\Big(f(t_k, y(t_k)) - f(t_k, y_k) \Big) + \frac{h^2}{2} y''(\xi_k)$$

Taking absolute values:
$$|d_{k+1}| \le |d_k| + h \left| f(t_k, y(t_k)) - f_k \right| + \frac{h^2}{2} |y''(\xi_k)|$$

$$\le |d_k| + h \underbrace{L |d_k|}_{\text{Lipshitz in } y:\ |f(t,x) - f(s,y)| \le L|x-y|} + \frac{h^2}{2} M \quad \max |y''|$$

$$= (1 + hL) |d_k| + \frac{h^2}{2} M$$

$\boxed{\text{Aside: If } \gamma_{k+1} \le (1+\alpha)\gamma_k + \beta \text{ then } \gamma_n \le e^{n\alpha}\gamma_0 + \frac{e^{n\alpha} - 1}{\alpha}\beta}$

$$\Rightarrow |d_k| \le e^{khL} |d_0| + \frac{e^{khL} - 1}{hL} \frac{h^2}{2} M$$
$$= e^{khL} |d_0| + \frac{e^{khL} - 1}{L} \frac{h}{2} M$$

Taking max's:

$$\max_{k:\, t_k \in [0,T]} |d_k| \le e^{L(T-t_0)} \underbrace{|d_0|}_{\substack{\to\, 0 \text{ as } h\to 0 \\ (=0)}} + \underbrace{\frac{e^{L(T-t_0)}-1}{L} \, \frac{M}{2}}_{\text{constant}} \overset{0 \text{ as } h\to 0}{h}$$

$\sim O(h)$ Local & global convergence

first order convergence.

Higher order analogues

Take more terms in the Taylor series:

$$y(t+h) = y(t) + h\,y'(t) + \frac{h^2}{2} y''(t) + O(h^3)$$

$$= y(t) + h\,f(t, y(t)) + \frac{h^2}{2} \frac{d}{dt}\Big(f(t, y(t))\Big) + O(h^3)$$

$$= y(t) + h\,f(t, y(t)) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right) + O(h^3)$$

$$= y(t) + hf(t, y(t)) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f\right) + O(h^3)$$

$\Rightarrow$ second-order Taylor method:

$$y_{k+1} = y_k + hf(t_k, y_k) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f\right)(t_k, y_k)$$

Local error: $O(h^2)$

$\boxed{\text{Higher order methods exist, require more derivatives of } f}$