Then are two types of ODEs:

**Initial value problems (IVPs):**

\[ (x) \quad y'(t) = f(y,t) \quad y(a) = y_0 \]

and boundary value problems (BVPs):

**Ex:** \[ ay'' + by' + cy = 0 \]

\[ y(0) = y_0, \quad y(1) = y_1 \]

The numerical methods for their solution are different, and we will focus on IVPs.

The exact solution to \((x)\) is given by

\[ y(t) = y_0 + \int_0^t f(y(\tau),\tau) \, d\tau \]

All numerical methods, in some form or another, approximate this exact solution.
The uniqueness of the solution to (*) is established by Piard's Theorem, see page 311 in the textbook. Basically, if \( f \) is continuous and bounded in a neighborhood of \((t_0, y_0)\), then a unique solution exists in some other neighborhood of \((t_0, y_0)\).

Another way of viewing numerical solutions to (*) is to directly approximate the differential part:

\[
y'(t) \approx \frac{y(t+h) - y(t)}{h} \approx f(y(t), t).
\]

\( \text{finite difference} \)

\[\Rightarrow \] Given \( y(t) \), to approximate \( y(t+h) \) we write

\[
y(t+h) - y(t) \approx f(y(t), t)
\]

\[\Rightarrow \]

\[
y(t+h) \approx y(t) + h f(y(t), t)
\]

\[\approx \int_{t}^{t+h} f(y(t), t) \, dt\]
This scheme is called the Forward Euler Method.

Let's turn to Finite Differences for a bit: approximating error and round-off error.

How good is the Forward Difference at approximating \( y'(t) \) in the presence of round-off error?

\[
\frac{y(t_{i+1}) - y(t_i)}{h} = \frac{y(t_{i+1})(1+\delta_1) - y(t_i)(1+\delta_2)}{h} \\
= \frac{y(t_{i+1}) - y(t_i)}{h} + \frac{y(t_{i+1}) \delta_1 + y(t_i) \delta_2}{h} \\
= \left( \frac{y(t_i) + h \cdot y'(t_i) + \frac{h^2}{2} y''(t_i)}{h} \right) - y(t_i) + \frac{y(t_{i+1}) \delta_1 + y(t_i) \delta_2}{h} \\
= y'(t_i) + \frac{h}{2} y''(t_i) + \frac{y(t_{i+1}) \delta_1 + y(t_i) \delta_2}{h}
\]

The error is given as \( Err(h) = y' - \frac{y(t_{i+1}) - y(t_i)}{h} \)
\[
\Rightarrow |\text{Err}| \leq \frac{h}{2} y''(\xi) + \left| \frac{y(t+h) s_1 + y(t) s_2}{h} \right|
\]

\[
\leq O(h) + \left( |y(t+h)| + |y(t)| \right) \frac{\varepsilon}{h}
\]

\[
\leq O(h) + O\left( \frac{\varepsilon}{h} \right)\text{ truncation error} + O\left( \frac{\varepsilon}{h} \right)\text{ round-off error}
\]

or approximate error

To minimize this error, \( h \) must be chosen to balance these terms:

\[
\frac{\varepsilon}{h} = \varepsilon \Rightarrow h = \sqrt{\varepsilon}
\]

If \( \varepsilon \approx 10^{-10} \), then choosing \( h \approx 10^{-5} \) minimizes the error, which is then of size \( 10^{-8} \).

That is the best you can hope for from a forward difference approximation.
An alternative approximation, the centered difference, has the property that

\[ |y'(t) - \frac{y(t+h) - y(t-h)}{2h}| \leq O(h^2) + O\left(\frac{h}{n}\right) \]

truncation error and step error

\[ \Rightarrow \text{choose } h^2 \sim \frac{6}{n} \text{ to balance terms} \]
\[ \Rightarrow h \sim \frac{3\sqrt{6}}{n} \text{ to minimize} \]

With \( h \sim 10^{-5} \), the total error above is \( \sim 10^{-10} \).

This is a 2nd-order accurate approximation.