Lecture 21 Numerical Analysis April 17, 2018

Last time:

- polynomial approximation in the 2-norm
  - analogous to least squares problems in finite dimensional linear alg.

- Minimize

$$\int_a^b |f(x) - p_n(x)|^2 w(x)\, dx$$

polynomial of deg. $\leq n$

- The solution is given as the orthogonal projection of f onto P_n ← the subspace of pols of deg $\leq n$.

- If $p_0, p_1, p_2 \ldots$ are orthonormal, the solution is

$$p_n = (f,p_0)\,p_0 + (f,p_1)\,p_1 + (f,p_2)\,p_2 + \ldots + (f,p_n)\,p_n .$$

Moving on: Numerical Integration

Almost no integral can be computed analytically, they must be evaluated numerically.

Ex.:

$$\underset{\text{error function}}{\mathrm{erf}(x)} = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

Application: ODEs, initial value problems

$$(*) \quad \begin{aligned} y'(t) &= f(t) \\ y(0) &= y_0 \end{aligned}$$

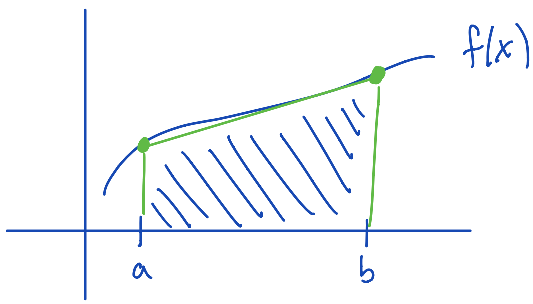
The analytic solution is

$$y(t) = y_0 + \int_0^t f(\tau)\, d\tau$$

All numerical methods for solving $(*)$ are based on approximating the integral in $(\cancel{\ast\ast\ast})$. More to come later.

The Trapezoidal (Trapezium) Rule

The most basic numerical integration technique:



Approximate f on $[a,b]$ by a line, integrate the line.

$$\int_a^b f(x)\,dx \approx \underbrace{\tfrac{1}{2}(b-a)\big(f(a)+f(b)\big)}_{\text{area for a trapezoid}}$$

This is a special case of more general integration formulae known as Newton-Cotes rules:

<u>Idea</u> Interpolate f on $[a,b]$, and integrate the interpolating polynomial.

This can be done with arbitrarily high degree, but remember: the interpolating polynomial may suffer from <u>Runge's Phenomenon</u>.

In general:

Interpolate f as

$$p_n(x) = \sum_{j=0}^{n} L_j(x)\, f(x_j)$$

Lagrange function:

$$L_k(x) = \prod_{i \neq k} \frac{x - x_i}{x_k - x_i}$$

Then the integral of f is approximated as:

$$\int_a^b f(x)\,dx \approx \int_a^b p_n(x)\,dx$$

$$= \int_a^b \sum_{j=0}^{n} L_j(x)\, f(x_j)\,dx$$

$$= \sum_{j=0}^{n} \left(\int_a^b L_j(x)\,dx \right) f(x_j)$$

$$= \sum_{j=0}^{n} w_j\, f(x_j)$$

quadrature weights

quadrature nodes (the locations where f is evaluated)