Lecture 20 Numerical Analysis April 12, 2018

Last time:

Approximation in $L^\infty[a,b]$

- in general, very hard, non-linear problem
- special case:
  the minimax polynomial approx to x^{n+1} is $x^{n+1} - \underbrace{\frac{1}{2^n} T_{n+1}(x)}_{\text{n^{th} deg. monic pol. with min ∞-norm.}}$.
- As a result of, one should always use the roots of T_n as interpolation nodes (when possible)

Approximation in the 2-norm

Recall from linear alg:

- the solution $\underline{x}$ to minimizing $\|A\underline{x} - \underline{b}\|_2$ is called the least squares solution.
- $\underline{x}$ is obtained by solving

$$A\underline{x} = QQ^T\underline{b} \quad \text{where} \quad Q^TQ = I \text{ and } A = QQ^TA.$$

orthogonal projection onto col A, obtained by the Gram-Schmidt process.

- To apply Gram-Schmidt, we need a vector space with an inner product (and therefore an induced norm). (This is called an inner product space).
- Trivially, P_n is a vector space.

We can define an inner product on P_n by:

$$(f,g) = \int_a^b f(x)\,g(x)\,dx$$

Check:

$$(f+g, h) = (f,h) + (g,h)$$

$$(cf, g) = c(f,g)$$

$$(f,g) = (g,f)$$

$$(f,f) > 0 \quad \text{iff} \quad f \neq 0.$$

Functions (polynomials) are orthogonal if $(f,g)=0$.

Let $p_0, p_1, \ldots, p_n$ be a basis for P_n. The the 2-norm approximation problem takes the form:

$$A = \min_{c_0,\ldots,c_n} \int_a^b \left| f(x) - \sum_{j=0}^n c_j p_j(x) \right|^2 dx$$

$$= \int_a^b |f(x)|^2 dx - 2 \sum_{j=0}^n c_j \int_a^b f(x) p_j(x)\, dx$$

$$+ \sum_{j=0}^n \sum_{k=0}^n c_j c_k \int_a^b p_j(x) p_k(x)\, dx$$

$$= (f,f) - 2 \sum_{j=0}^n c_j (f, p_j) + \sum_{j=0}^n \sum_{k=0}^n c_j c_k (p_j, p_k)$$

Writing down

$$\nabla A = \underline{0}, \quad \text{we have the}$$

equations

$$\frac{\partial A}{\partial c_0} = 0 \quad \cdots \quad \frac{\partial A}{\partial c_n} = 0$$

$$\Rightarrow \quad \frac{\partial A}{\partial c_\ell} = -2(f, p_\ell) + 2\sum_{k=0}^{n} c_k (p_\ell, p_k)$$

$$\Rightarrow \quad \sum_{k=0}^{n} c_k (p_\ell, p_k) = (f, p_\ell)$$

$(n+1) \times (n+1)$ matrix equation for $c_0 \ldots c_k$

$$\begin{pmatrix} (p_0, p_0) & \cdots & (p_0, p_n) \\ (p_1, p_0) & & \vdots \\ \vdots & & \vdots \\ (p_n, p_0) & & (p_n, p_n) \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} (f, p_0) \\ \vdots \\ (f, p_n) \end{pmatrix}$$

Therefore, the best 2-norm approx to f is

$$\varphi = c_0 p_0 + \ldots + c_n p_n.$$

Imagine now that $(p_j, p_k) = \delta_{jk}$ (orthonormal),

then directly $c_j = (f, p_j)$!

Def: If the sequence of polynomials $\varphi_0, \varphi_1, \varphi_2, \dots$, with $\deg \varphi_n = n$, on the interval $[a,b]$ satisfies

$$\int_a^b \varphi_j(x)\, \varphi_k(x) = 0 \quad \text{if} \quad j \neq k,$$

then $\varphi_0, \varphi_1 \dots$ is a <u>system of orthogonal polynomials</u>.

This definition can be extended with a weight function $w > 0$ on (a,b), since $(f,g)_w = \int f(x)\, g(x)\, w(x)\, dx$ defines an inner product.

In this case, we require that

$$\int_a^b \varphi_j(x)\, \varphi_k(x)\, w(x)\, dx = 0 \quad \text{if} \quad j \neq k.$$

Ex: Find $\varphi_0, \varphi_1, \varphi_2$ on $[-1,1]$ with weight function $w = 1$.

Set $\varphi_0 = 1$.

$\varphi_1 = ax + b$. $\quad \int_{-1}^{1} (ax+b) \cdot 1 \, dx = 0$

$\Rightarrow 2b = 0$

$= x \qquad \Rightarrow b = 0$

$$\varphi_2 = x^2 + bx + c$$

Two conditions must be satisfied:

$$\int_{-1}^{1} (x^2+bx+c)\cdot 1 \, dx = 0 \qquad \int_{-1}^{1} (x^2+bx+c)\cdot x \, dx = 0$$

$$\Rightarrow \frac{2}{3} + 2c = 0 \qquad \frac{2}{3}b = 0 \Rightarrow b = 0$$

$$\Rightarrow c = -1/3$$

$$\Rightarrow \varphi_2 = x^2 - \frac{1}{3} = \frac{1}{3}(3x^2 - 1)$$

These functions $\varphi_0, \ldots$, are known as Legendre polynomials. They form an orthogonal basis of $L^2[-1,1]$ under the inner product $(f,g) = \int_{-1}^{1} f(x) g(x) \, dx$.

[Show that $L^2(a,b)$ is a vector space.]

Legendre polynomials can be constructed another way as well: the Gram-Schmidt process.

Start with $\left.\begin{aligned} \varphi_0 &= 1. \\ \varphi_1 &= x. \end{aligned}\right\}$ automatically orthogonal.

set $m_2(x) = x^2$ $\leftarrow$ lin. indep. from φ_0, φ_1.

Compute $\varphi_2 = m_2 - \mathrm{Proj}_{\{\varphi_0, \varphi_1\}} m_2$

$$= m_2 - \frac{(m_2, \varphi_0)}{(\varphi_0, \varphi_0)} \varphi_0 - \frac{(m_2, \varphi_1)}{(\varphi_1, \varphi_1)} \varphi_1$$

$$= x^2 - \frac{2}{3}\frac{1}{2}\cdot 1 - 0\cdot x$$

$$= x^2 - \frac{1}{3}$$

In general, compute

$$\varphi_n = m_n - \sum_{\ell=0}^{n-1} \frac{(m_n, \varphi_\ell)}{(\varphi_\ell, \varphi_\ell)} \varphi_\ell$$

$$= x^n - \sum_{\ell=0}^{n-1} \frac{\int x^n \varphi_\ell(x)\,dx}{\underbrace{\int \varphi_\ell(x)^2\,dx}_{\|\varphi_\ell\|^2}} \varphi_\ell(x)$$

These polynomials can be scaled to any interval.

If $\int_{-1}^{1} \varphi_j(x)\varphi_k(x)\,dx = 0$ if $j \neq k$, then

$$\int_a^b \varphi_j(t)\,\varphi_k(t)\,dt = 0 \text{ if } j \neq k \text{ with } t = \left(\frac{x+1}{2}\right)(b-a) + a$$

<u>Ex:</u> Chebyshev polynomials:

We know that $\int_0^{\pi} \cos mt \cos nt \,dt = 0$ if $m \neq n$.

Let $t = a\cos x$, $dt = \frac{1}{\sqrt{1-x^2}}\,dx$

$$\Rightarrow \int_{-1}^{1} \cos(m \operatorname{acos} x) \cdot \cos(n \operatorname{acos} x) \frac{dx}{\sqrt{1-x^2}}$$

$$= \int_{-1}^{1} T_m(x)\, T_n(x) \frac{dx}{\sqrt{1-x^2}}$$

Therefore, $T_0, T_1, \ldots$ are orthogonal with respect to the weight function $\frac{1}{\sqrt{1-x^2}}$.

<u>Thm</u> For $f \in L^2_w(a,b)$, there is a unique $p_n \in P_n$ such that

$$\|f-p_n\|_{w,2} = \min_{q \in P_n} \|f-q\|_{w,2}.$$

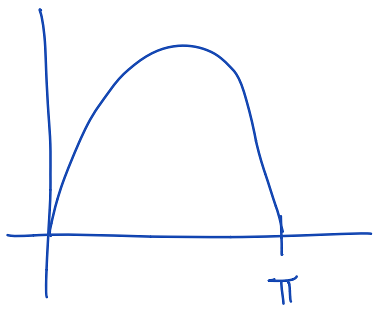
<u>Proof</u>: Gram-Schmidt, solve directly for the coefficients in the resulting approximation.

(Note: This is just linear algebra!).

Famous sets of orthogonal polynomials:

$w(x)$	(a,b)	Pols
1	$(-1,1)$	Legendre
$1/\sqrt{1-x^2}$	$(-1,1)$	Chebyshev
e^{-x}	$(0,\infty)$	Laguerre
e^{-x^2}	$(-\infty,\infty)$	Hermite.

<u>Ex:</u> Compute best quadratic 2-norm approx to $f(x) = \sin x$ on $(0, \pi)$ with $w=1$.



$\Rightarrow$ Legendre polynomials

The Legendre polynomials on $[-1,1]$ are

$$\varphi_0 = 1 \quad , \quad \varphi_1 = x \quad , \quad \varphi_2 = \frac{1}{3}(x^2 - 1)$$

To define on $[0,\pi]$, let $t = \frac{x+1}{2}\pi$,

and so $\Rightarrow x = \frac{2t}{\pi} - 1$

$$\tilde{\varphi}_0 = 1 \quad , \quad \tilde{\varphi}_1 = \frac{2t}{\pi} - 1 \quad , \quad \tilde{\varphi}_2 = \frac{1}{3}\left(\left(\frac{2t}{\pi} - 1\right)^2 - 1 \right)$$

$$= \frac{2}{\pi}\left(t - \frac{\pi}{2}\right) \qquad\qquad = \frac{4t}{3\pi^2}\left(t - \pi\right)$$

$\Rightarrow$ The best approx is

$$p_2(t) = \frac{(f, \tilde{\varphi}_0)}{(\tilde{\varphi}_0, \tilde{\varphi}_0)} \tilde{\varphi}_0 + \frac{(f, \tilde{\varphi}_1)}{(\tilde{\varphi}_1, \tilde{\varphi}_1)} \tilde{\varphi}_1 + \ldots$$

$$\int \sin x \; x \, dx = \sin x - x \cos x$$

$$\int \sin x \; x^2 \, dx = 2x \sin x - (x^2 - 2) \cos x$$

Relationship with differential operators:

Sturm-Liouville operator:

$$Lu = -(pu')' + qu$$

It is linear, and self-adjoint

$\Rightarrow$ eigenvalues are real

$$Lu = \lambda w u$$

weight function

$\Rightarrow$ eigenfunctions are orthogonal under

$$(f, g) = \int f(x) g(x) w(x) \, dx$$

Applications in numerical analysis, quantum physics, etc.