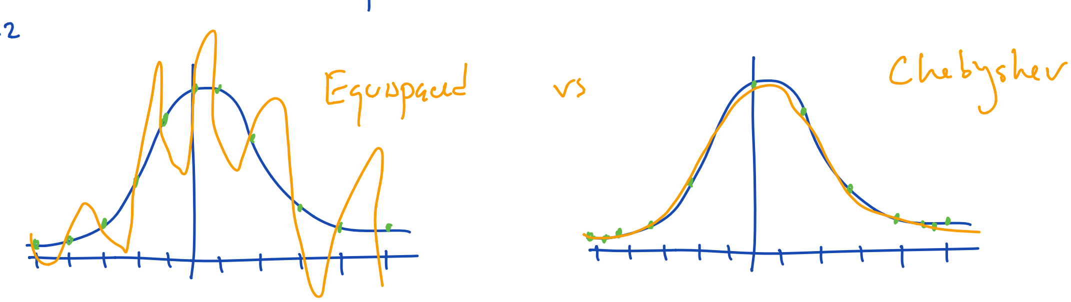
Lecture 19 Numerical Analysis 4/10/18

Last time:

- Barycentric forms of Lagrange interpolation
    - obtained by simply rewriting the standard Lagrange form
    - superior from a numerical stability point of view.

- Convergence of the interpolating polynomial as $n \to \infty$ (degree, # nodes)
    - For poorly chosen nodes, Runge's phenomenon is encountered

$\frac{1}{1+x^2}$



What is the reason for this behavior? It is related to the fact that $f(x) = \frac{1}{1+x^2}$ has a singularity at $x = \pm i$ in the complex plane.

This dictates the radius of convergence of its Taylor series. $\left(1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots\right)$

Using "Chebyshev points" sums to fix this problem.... Details to follow.

Function Approximation

Polynomial interpolation mainly has application in function approximation, with respect to some norm. For functions:

$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|.$$

$$\|f\|_2 = \sqrt{\int_a^b |f(x)|^2 \, dx}$$

$$\|f\|_1 = \int_a^b |f(x)| \, dx$$

just like for vectors.

Norms of functions satisfy the same properties as those in the finite dimensional vector case:

① $\|f\| \geq 0$, $\|f\|=0$ iff $f=0$

② $\|cf\| = |c| \, \|f\|$

③ $\|f+g\| \leq \|f\| + \|g\|$

Ex: The 2-norm can be generalized with a "weight function" $w>0$:

$$\|f\|_{2,w} = \sqrt{\int_a^b |f(x)|^2 w(x)\, dx}$$

(this is the analogue of defining $\|\underline{x}\| = (\underline{x}, A\underline{x})$, where A is symmetric positive def.)

So: the polynomial p_n of degree n that best approximates a function f in the ∞-norm is

$$\min_{p_n \in P_n} \underbrace{\|p_n - f\|_\infty}$$

the ∞-norm of the difference, i.e. maximum pointwise error.

From Analysis class, we know that continuous functions f on an interval can be approximated arbitrarily closely by a polynomial of "some" degree: this known as the Weierstrass Approximation Theorem.

I.e. For any $\epsilon > 0$, there exists a polynomial p such that

$$\|f-p\|_\infty < \epsilon \quad , \quad \|f-p\|_2 < \infty \quad \left(\text{maybe different } p\text{'s}\right).$$

This is a completely useless theorem for numerical applications.

The question of restricting $p_n \in P_n$ is much more interesting, and actually useful.

For $n > 0$, find $p_n \in P_n$ such that

$$\|f-p_n\|_\infty = \min_{q \in P_n} \|f-q\|_\infty$$

Thm Such a p_n exists, and is unique. (The proof does not tell us how to find p_n.)

In general, one cannot write down the <u>minimax</u> polynomial, i.e. the p_n such that

$$\| f - p_n \|_\infty = \min_{q \in P_n} \max_{x \in [a,b]} | f(x) - p_n(x) |$$

<u>However:</u> We can explicitly write the minimax polynomial approx to the monomial x^{n+1}:

<u>Thm:</u> Let $n \geq 0$, then

$$\| p_n(x) - x^{n+1} \|_\infty \text{ is minimized when}$$

$$p_n(x) = x^{n+1} - \frac{1}{2^n} \cos\left((n+1) \operatorname{acos} x \right).$$

The function $T_n(x) = \cos(n \operatorname{acos} x)$ is known as the Chebyshev polynomial of degree n. They play a very important role in numerical analysis.

Chebyshev Polynomials

$$T_n(x) = \cos(n \operatorname{acos} x) \qquad n = 0, \ldots$$

$T_0(x) = 1$ (usually only concerned with $x \in [-1,1]$)

$T_1(x) = x$

$\vdots$

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$$ ← polynomial of degree $n+1$.

Trivially, the zeros of T_n can be computed as

$$\cos(n \operatorname{acos} x) = 0$$

$$\Rightarrow \quad n \operatorname{acos} x = \frac{\pi}{2}(2m+1)$$

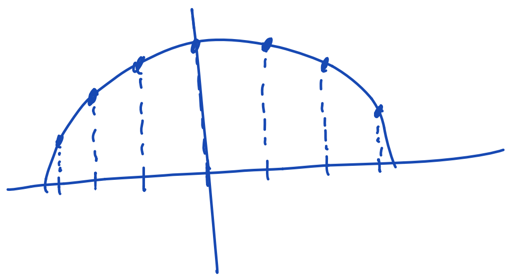
$$\operatorname{acos} x = \frac{\pi}{2n}(2m+1)$$

$$x = \cos\left(\frac{(2m+1)\pi}{2n}\right) \qquad m = 0 \ldots$$ repeats after $m > n$

The roots on $[-1,1]$, ordered, are:

$$x_j = \cos\left(\frac{(2j-1)\pi}{2n}\right) \qquad j = 1, \ldots, n \quad (n \text{ roots}).$$

These points are the projection onto the x-axis of equispaced points on the unit circle:



Claim: Interpolation of a function f on $[a,b]$ with a degree n polynomial p_n at the n Chebyshev nodes yields a near minimax polynomial approximate.

Idea: The approximation error of the interpolation is

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j)$$

If x_j are chosen to be roots of T_n (properly scaled to $[a,b]$),

then $$\prod_{j=0}^{n} (x - x_j) = \frac{1}{2^n} T_n(x).$$

It can be shown that $\frac{1}{2^n} T_n$ is the minimum norm monic polynomial.

Approximation in the 2-norm

The 2-norm of a function, with general continuous weight function $w > 0$, on (a,b) is:

$$\|f\|_2^2 = \int_a^b |f(x)|^2 w(x)\, dx .$$

We want to find p_n such that

$\|f - p_n\|_2$ is minimized. This is a least squares approximation to f – integration plays the rol of $\sum r_i^2$ here.

Therefore, the solution is obtaind by computing the orthogonal projection of f onto the space P_n, under the inner product $(f,g) = \int_a^b f(x)\, g(x)\, w(x)\, dx.$

There is no reason why the best p_n for the 2-norm error is the same p_n for the ∞-norm error.

Lets look at this in some more detail.

<u>Our goal</u>: For a function $f \in L^2_w(a,b)$ find $p_n \in P_n$ such that:

$$\| f - p_n \|_2^2 = \inf_{q \in P_n} \| f - q \|_2^2$$

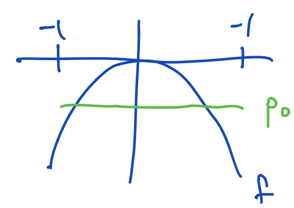
$$= \inf_{q \in P_n} \int | f(x) - q(x) |^2 w(x)\, dx.$$

<u>Ex</u>: Let $n=0$, $f(x) = -2x^2$ on $[-1,1]$, $w=1$.

Find $p_0 = c$ to minimize $\| f - p_0 \|_2^2$.

$$\| f - p_0 \|_2^2 = \int_{-1}^{1} | -2x^2 - c |^2 \, dx$$

$$= 2c^2 + \frac{8c}{3} + \frac{8}{5}$$



$$\frac{d}{dc} \| e \|_2^2 = 4c + \frac{8}{3} = 0 \quad \Rightarrow \quad c = -\frac{2}{3}$$

Contrast with the $\| \cdot \|_\infty$ approximation, which would be $p_0 = -1$.