Lecture 18 Numerical Analysis 4/5/18

Last time

- Lagrange interpolation

For data (x_j, y_j), $j = 0, \dots, n$

there is a unique polynomial interpolant p_n of degree n such that:

$$p_n(x_j) = y_j$$

An explicit construction of p_n is the Lagrange form:

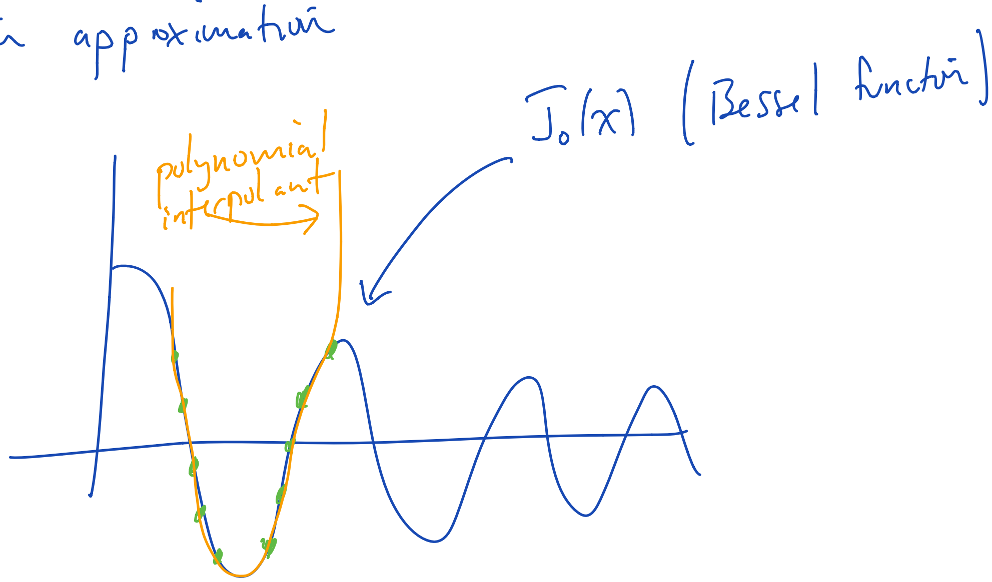
$$p_n(x) = \sum_{k=0}^{n} L_k(x)\, y_k(x)$$

with $$L_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)}$$

L_k has the property that

$$L_k(x_k) = 1$$
$$L_k(x_j) = 0 \quad \text{if } j \neq k.$$

Polynomial interpolation is often used for function approximation



Error in polynomial interp is given by:

<u>Thm</u>: Let $f \in C^{n+1}[a,b]$. For $x \in [a,b]$, there exists $\xi = \xi(x) \in (a,b)$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \underbrace{\prod_{j=0}^{n} (x - x_j)}_{\text{depends on the choice of points.}}$$

This is a property of polynomials, <u>not</u> the particular Lagrange form (since the polynomial is unique).

To answer our 3^{rd} question last time:

(3) In floating point arithmetic, is the evaluation of p_n stable?

$\boxed{\text{Question 3}}$ The numerical stability of evaluating p_n in Lagrange form.

Short story: The basic Lagrange form can be unstable (overflow, underflow, round-off error, etc.)

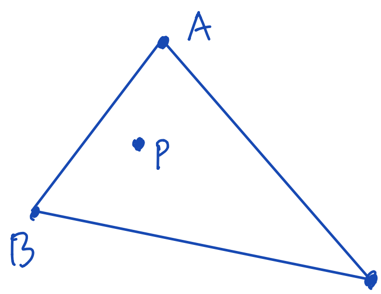
(skip analysis, tedious, ...)

The Barycentric Form(s) of Interpolation

The numerical stability of evaluating an interpolating polynomial can be fixed by rearranging its terms.

As motivation, examine barycentric coordinates on a triangle:

Ex: The barycentric coordinates of a point P inside a triangle with vertices A, B, C is given by:



$(\alpha, \beta, \gamma) \leftarrow$ Barycentric coordinates

with $P = \alpha A + \beta B + \gamma C$

and $\alpha + \beta + \gamma = 1$.

I.E. "center" of triangle is given by

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$ (center of mass).

<u>Idea</u> Replace A, B, C with <u>functions</u> that sum to 1.

Start with Lagrange form, and rewrite

$$P_n(x) = \sum_{k=0}^{n} \left(\prod_{j \neq k} \frac{(x - x_j)}{(x_k - x_j)} \right) y_k$$

$$= \sum_{k=0}^{n} \left(\prod_{j=0}^{n} (x - x_j) \right) \frac{1}{x - x_k} \left(\prod_{j \neq k} \frac{1}{(x_k - x_j)} \right) y_k$$

$$= \varphi(x) \sum_{k=0}^{n} \frac{w_k}{(x - x_k)} y_k$$

(Modified Lagrange)
First Barycentric Formula

Stability analyzed in 2004 $\Rightarrow$ backward stable. (Higham, 2004)

when $\varphi(x) = \prod_{j=0}^{n} (x - x_j)$, $w_k = \prod_{j \neq k} \frac{1}{x_k - x_j}$

This modified Lagrange form can be further simplified by "dividing by 1":

The polynomial interpolant of the function $\underline{1}$ at the same points (x_j, y_j) is simply:

$$\underline{1} = \varphi(x) \sum_{k=0}^{n} \frac{w_k}{(x - x_k)} \qquad (\text{since } y_k = 1).$$

Then computing

$$p_n(x) = \frac{\varphi(x) \sum_{k=0}^{n} \frac{w_k}{x - x_k} y_k}{\underline{1}} = \frac{\varphi(x) \sum_{k=0}^{n} \frac{w_k}{x - x_k} y_k}{\varphi(x) \sum_{k=0}^{n} \frac{w_k}{x - x_k}} = \frac{\sum_{k=0}^{n} \frac{w_k}{x - x_k} y_k}{\sum_{k=0}^{n} \frac{w_k}{x - x_k}}$$

Second Barycentric Formula.

This form is "forward stable for any reasonable choice of x_j". Basically, the errors in the top cancel out the errors in the bottom (i.e. exactly when $x \approx x_k$). (Also see Higham 2004).

This is the form that should almost always be used.

Cost ?

Computing w_k : $O(n)$

all w_k's : $O(n^2)$.   only fair since solving for actual coefficients is $O(n^3)$.

Updating w_k's with new data : $O(n)$

Evaluating $p_n(x)$: $O(n)$.

Convergence of Polynomial Interpolation

Let's examine the question of what happens as $n \to \infty$, i.e.

$$\lim_{n\to\infty} \max_x |f(x) - p_n(x)| = ? \quad \Big(\text{here, } \infty\text{-norm} \Big).$$

the pointwise error $\sim \max_\xi \dfrac{|f^{(n+1)}(\xi)|}{(n+1)!} \cdot \max_x \prod |x - x_j|$

It's not obvious if this decreases or increases...

(Show Matlab demo).

For equispaced and randomly distributed points, the phenomenon you are witnessing is known as the <u>Runge effect</u> $\left(\frac{1}{1+x^2} \text{ is the Runge function} \right)$.