Lecture 16 — Numerical Analysis — March 29, 2018

Last time

Methods for finding a single $\lambda, \underline{v}$ pair

$$A\underline{v} = \lambda v$$

1. Power Method
2. Power Method with shift
3. Inverse Power Method With Shift

If $A\underline{v} = \lambda \underline{v}$, then $(A - sI)^{-1}\underline{v} = \underbrace{\frac{1}{\lambda - s}}\underline{v}$

with proper choices of s, this eigenpair can be made to dominate the spectrum of $(A-sI)^{-1}$

$\Rightarrow$ Each iteration requires a "solve", or computation of $(A-sI)^{-1}$ explicitly

Jacobi's Method

The power method and inverse Power method only compute eigenpairs one at a time.

Note: The matrices A and $\underbrace{B = M^{-1}AM}_{\text{similarity transform.}}$ have the same eigenvalues.

Pf: Look at their characteristic polynomials.

If M is chosen to be the matrix of eigenvectors, then B is diagonal with λ_j on the diagonal.

Ex: Let A be a real symmetric 2×2 matrix.

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

Then its eigenvalues are real, and it is diagonalized by an orthogonal matrix V.

All 2×2 orthogonal matrices can be parameterized as:

$$V = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \quad \begin{pmatrix} 2\times2 \text{ rotation} \\ \text{matrix} \end{pmatrix}.$$

Then we want

$$V^T A V = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

The two equations for the 12 and 21 entries are:

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \underbrace{\begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

$$\underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} a\cos\varphi - b\sin\varphi & a\sin\varphi + b\cos\varphi \\ b\cos\varphi - d\sin\varphi & b\sin\varphi + d\cos\varphi \end{pmatrix}}$$

$$\Rightarrow \quad a\cos\varphi\sin\varphi + b\cos^2\varphi - b\sin^2\varphi - d\cos\varphi\sin\varphi = 0$$

$$a\cos\varphi\sin\varphi - b\sin^2\varphi + b\cos^2\varphi - d\cos\varphi\sin\varphi = 0 \quad \checkmark$$

How do we find φ?

$$(a-d)\cos\varphi\,\sin\varphi + b\left(\cos^2\varphi - \sin^2\varphi\right) = 0$$

$$\Rightarrow \quad (a-d)\,\frac{1}{2}\sin 2\varphi + b\cos 2\varphi = 0$$

$$\Rightarrow \quad \tan 2\varphi = \frac{2b}{d-a}$$

$$\Rightarrow \quad \varphi = \frac{1}{2}\,\mathrm{atan}\left(\frac{2b}{d-a}\right)$$

(in Fortran / C use the function atan2(y,x).)

The Jacobi Method

Define

$$R^{pq}(\varphi) = \begin{array}{r} \\ \\ \text{row } p \rightarrow \\ \\ \text{row } q \rightarrow \\ \\ \\ \end{array} \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && \cos\varphi &&& \sin\varphi & \\ &&& 1 \ddots 1 &&& \\ && -\sin\varphi &&& \cos\varphi & \\ &&&&&& 1 \ddots \\ &&&&&&& 1 \end{pmatrix}$$

$R^{pq}(\varphi)$ can be used to set the pq and qp elements of real symmetric A to zero

① Set $A^{(0)} = A$

② Find pq element in $A^{(k)}$ with maximum abs.

③ Compute $\varphi_k = \frac{1}{2} \operatorname{atan}\left(\frac{2 a_{pq}^{(k)}}{a_{qq}^{(k)} - a_{pp}^{(k)}} \right)$

④ Set $A^{(k+1)} = R^{pq}(\varphi_k)^T A^{(k)} R^{pq}(\varphi_k)$

Continue until all $|a_{pq}^{(k)}| < \epsilon$, $p \neq q$.

Then $A^{(k)} \to \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$$R^{(k)} = R(\varphi_1) \cdots R(\varphi_k) \to (\hat{\underline{v}}_1 \cdots \hat{\underline{v}}_n)$$

~~[illegible]~~

What can we say about convergence of Jacobi's Method?

Basically, we want to show that the off-diagonal elements go to zero. First, a Lemma.

Lemma: If R is an orthogonal transform,

then $\|A\|_F = \|R^T A R\|_F$.

Frobenius norm.

$$\|A\|_F = \left(\sum_i \sum_j |a_{ij}|^2\right)^{1/2}.$$

Pf: ~~If A and B~~

If $B = R^T A R$, then A and B have the same eigenvalues. $B^2 = (R^T A R)(R^T A R) = R^T A^2 R$.

So A^2 and B^2 have the same eigenvalues, therefore

$\operatorname{trace}(A^2) = \operatorname{trace}(B^2)$.

But $\|A\|_F^2 = \operatorname{trace}(A^T A) = \operatorname{trace}(A^2) = \operatorname{trace}(B^2) = \|B\|_F^2$.

Next, let's split the Frobenius norm into ~~3~~ 2 pieces:

call: $\|A\|_F^2 = S(A) = \sum_{i,j} |a_{ij}|^2$

$D(A) = \sum_i |a_{ii}|^2$ diagonal part

$L(A) = \sum_{i \neq j} |a_{ij}|^2$ off diagonal part

$\Rightarrow S(A) = D(A) + L(A) = \|A\|_F^2$.

<u>Theorem</u> Let $A^{(k)}$ be the iterats in the classical Jacobi method. Then

$$\lim_{k\to\infty} L(A^{(k)}) = 0$$

$$\lim_{k\to\infty} D(A^{(k)}) = \text{trace}(A^2).$$

<u>Proof</u>: Let a_{pq} be element of A with largst <u>abs</u>.

Let $B = R^{pq}(\varphi)^T A R^{pq}(\varphi)$.

Then
$$\begin{pmatrix} b_{pp} & b_{pq} \\ b_{qp} & b_{qq} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}^T \begin{pmatrix} a_{pp} & a_{pq} \\ a_{qp} & a_{qq} \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

and $b_{pq} = b_{qp} = 0$.

But from Lemma, $b_{pp}^2 + b_{qq}^2 = a_{pp}^2 + a_{qq}^2 + 2a_{pq}$.

Furthermore,
$$\begin{aligned} S(B) &= D(B) + L(B) \\ &= S(A) \\ &= D(A) + L(A) \end{aligned}$$

B has the same diagonal elements of A, except for b_{pp} and b_{qq}, so $D(B) = D(A) + 2a_{pq}^2$

$$\Rightarrow L(B) = L(A) - 2a_{pq}^2.$$

Since a_{pq} was the largest off diagonal element of A, $L(A) \le n(n-1)\, a_{pq}^2$.

$$\Leftrightarrow \quad a_{pq}^2 \ge \frac{L(A)}{n(n-1)}.$$

Therefore,
$$\begin{aligned} L(B) &= L(A) - 2a_{pq}^2 \\ &\le L(A) - 2\frac{L(A)}{n(n-1)} \\ &= L(A)\left(1 - \frac{2}{n(n-1)}\right) \end{aligned}$$

Re-labeling, $\begin{matrix} A^{(1)} = B \\ A^{(0)} = A \end{matrix}$, we have that

$$L(A^{(1)}) \le \underbrace{\left(1 - \frac{2}{n(n-1)}\right)}_{<1} L(A^{(0)})$$

$$\Rightarrow \quad L(A^{(k+1)}) \le \left(1 - \frac{2}{n(n-1)}\right)^k L(A^{(0)})$$

$$\to 0 \text{ as } k \to \infty.$$

And since
$$\begin{aligned} S(A^{(k)}) &= D(A^{(k)}) + L(A^{(k)}) \\ &= \text{trace}(A^2), \end{aligned}$$

$L(A^{(k)}) \to 0$ we have that

$$\lim_{k\to\infty} D(A^{(k)}) = \text{trace}(A^2).$$

What guarantes that

$$\lim_{k\to\infty} D(A^{(k)}) = \operatorname{trace}(A^2)$$

implies that $A^{(k)} \to \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$?

<u>Idea</u> Apply the Gerschgorin Theorm.

~~Since~~ $A^{(k)}$ and A have the same eigenvalues, and $L(A^{(k)}) \to 0$, the Gerschgorin disks of $A^{(k)}$ have radii going to 0 as well. Therefore, the eigenvalues of $A^{(k)}$, and therefore A, are ~~on~~ the limit of the diagonal of $A^{(k)}$.