First Eigenvalues & Eigenvectors

Recall:

$\lambda, \vec{v}$ is an eigenvalue & corresponding eigenvector of an $n \times n$ matrix A if:

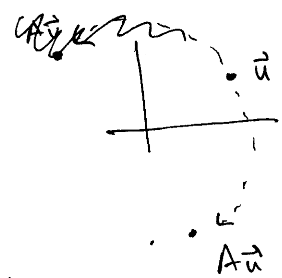
$$A\vec{v} = \lambda \vec{v}$$

$$\Rightarrow (A - \lambda I)\vec{v} = 0$$

$$\Rightarrow \det(A - \lambda I) = 0$$ ← this is a polynomial in λ, known as the characteristic polynomial $\chi(\lambda)$.

Eigenvalues of real matrix may be complex:

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ →



rotation.

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\chi(\lambda) = \lambda^2 + 1$$

$\Rightarrow \lambda = \pm i$ ← complex conjugate.

$\Rightarrow \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

complex conjugates.

$\Rightarrow \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \vec{v}_2 \Rightarrow \vec{v}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

Since If A is real, $A\vec{v}_1 = \lambda_1 \vec{v}_1$

$\Rightarrow A\bar{\vec{v}}_1 = \bar{\lambda}_1 \bar{\vec{v}}_1$

If $\vec{v}_1$ is eigenvector, so is $\bar{\vec{v}}_1$.

~~How do we find eigenvalues / eigenvectors without solving polynomial equations?~~

<u>Multiplicities</u> λ may be a repeated root of $\chi(\lambda)$ $\rightarrow$ then find basis for the <u>eigenspace</u>. The dimension of the eigenspace may or may not be the same as the multiplicity of λ:

<u>Ex:</u> $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ (shear matrix)

$$\chi(\lambda) = (1-\lambda)(1-\lambda) = 0 \Rightarrow \lambda = 1$$

$$A - \lambda I = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \Rightarrow (A - \lambda I)\vec{v} = 0$$

$$\Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ only one eigenvector.

<u>Diagonalizable Matrices</u> If A has n ~~distinct~~ linearly independent eigenvectors, then it is <u>diagonalizable</u>:

Let $V = \underbrace{(\vec{v}_1 \cdots \vec{v}_n)}$. Then $AV = V\Lambda$, and since it is invertible, $A = V\Lambda V^{-1}$, or likewise:

$$\Lambda = V^{-1}AV \leftarrow \underbrace{\text{similarity transformation}}_{\text{preserves eigenvalues.}}$$

V^{-1}: rows are left eigenvectors

V: columns are right eigenvectors

Best almost case for numerical Linear algebra: A is real-symmetric

(best case would be real symmetric positive definite).

$\Rightarrow \quad A^t = A, \quad A^* = A.$

$\Rightarrow$ Eigenvalues are real, and A is diagonalized by an orthogonal matrix: Q.

~~Pf: Let $\lambda, \vec{v}$ be eigenvalue/vector pair.~~

$\cancel{A\vec{v} = \lambda\vec{v}}$ $\quad \cancel{\vec{v}^t A \vec{v} = \lambda \vec{v}^t \vec{v}}$

$\cancel{A^t\vec{v} = \lambda\vec{v}.}$

Pf: Let $\vec{u}, \vec{v}$ correspond to two eigenvals, λ_1, λ_2.

Then $(A\vec{u}, \vec{v}) = (\lambda_1 \vec{u}, \vec{v}) = \lambda_1 (\vec{u}, \vec{v})$

but also $(A\vec{u}, \vec{v}) = (\vec{u}, A\vec{v}) = \lambda_2 (\vec{u}, \vec{v})$

$\Rightarrow (A\vec{u}, \vec{v}) - (A\vec{u}, \vec{v}) = 0 = (\lambda_1 - \lambda_2)(\vec{u}, \vec{v})$

$\Rightarrow (\vec{u}, \vec{v}) = 0$ since $\lambda_1 \neq \lambda_2$ (by assumption).

What if A does not have n lin. indep. eigenvectors?

$\Rightarrow$ Jordan form instead of diagonal form.

What we mean is:

Instead of A being similar to a diagonal matrix, $W^{-1}AW = D$, A.

is similar to one in Jordan form:

For some W, $W^{-1}AW = J$, with

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix}, \quad J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$$

- # blocks = # linearly independent eigenvectors

~~size of block~~

[This form is unstable for small changes in the entries of A or J.

In general, what can we say about the eigenvalues of a particular matrix? How do we calculate them? (why do we care?)

Ex: ODE IVP:

$\vec{y}\,' = A\vec{y}$ is equivalent (under a change of variables) to the system:

$$\vec{u}\,' = D\vec{u}$$

D diagonal: $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ eigenvalues of A.

<u>Gerschgorin's Theorem</u> Let A be an $n \times n$ matrix (real or complex). Then all the eigenvalues of A lie in $\bigcup_i D_i$, with

$$D_i = \{ z \in \mathbb{C} \text{ s.t. } |z - a_{ii}| \le \underbrace{\sum_{j \ne i} |a_{ij}|}_{\text{sum of off-diagonal entries of row } i.} \}$$

(D_i is a "Gerschgorin Disc")

If m discs are connected, then m eigenvalues are located in this region.

<u>Pf</u>: Let $\lambda, \vec{v}$ be an eigenvalue/vector pair.

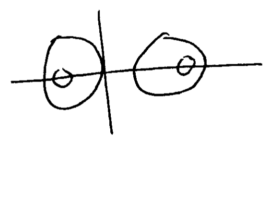
Then $A\vec{v} = \lambda \vec{v} \Rightarrow \sum a_{ij} v_j = \lambda v_i$ (for all i)

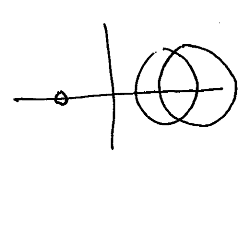
$$\Rightarrow (\lambda - a_{ii}) v_i = \sum_{j \ne i} a_{ij} v_j$$

pick largest entry of $\vec{v}$, v_k $\Rightarrow$ $(\lambda - a_{kk}) v_k = \sum_{j \ne k} a_{kj} v_j$

$$|\lambda - a_{kk}| \le \sum_{j \ne k} |a_{kj}| \left| \frac{v_j}{v_k} \right| \le \sum_{j \ne k} |a_{kj}|$$

<u>The trick you can play:</u>

A: 

$W^{-1}AW$: 

The power Method

Goal: Calculate the largest eigenpair

(Assume A is diagonalizable)

Start with random vector $\vec{w} = \sum c_j \vec{v}_j$

Then $\quad A\vec{w} = \sum c_j A\vec{v}_j = \sum c_j \lambda_j \vec{v}_j$

$$\Rightarrow A^k\vec{w} = \sum c_j \lambda_j^k \vec{v}_j \approx c_j \lambda_j^k \vec{v}_j$$ for sufficiently large k.

If $\lambda_1 > \lambda_2$ (sufficiently) then λ_1 dominates.

Normalize on each step:

$$\vec{w}_0 = \vec{w}/\|\vec{w}\|$$

$$\vec{w}_k = \frac{A\vec{w}_{k-1}}{\|A\vec{w}_{k-1}\|} \longrightarrow \text{eigenvector.}$$

To calculate λ_1,

1. $w_{ki}/w_{(k-1)i}$ (but could be small)

2. set $\lambda_1^{(k)} = (A\vec{w}_{k-1}, \vec{w}_{k-1})$ (since $\|\vec{w}_{k-1}\| = 1$)

The rate of convergence is roughly:

$$\|\vec{v}^{(k)} - \vec{v}\| \approx O\left(\left(\frac{\lambda_2}{\lambda_1}\right)^k\right)$$