What does this have to do with the matrix condition number?

By our definition earlier: sensitivity of the solution to the input

$$\underset{\underline{x}}{\uparrow} \qquad\qquad \underset{\underline{b}}{\uparrow}$$

Let $\|\underline{b}-\hat{\underline{b}}'\|$ be small, and $\underline{x} = A^{-1}\underline{b}$, $\underline{x}' = A^{-1}\underline{b}'$.

Then $\|\underline{x}-\underline{x}'\| = \|A^{-1}\underline{b} - A^{-1}\underline{b}'\|$

$$\leq \underbrace{\|A^{-1}\|}_{\text{absolute condition number}} \|\hat{\underline{b}} - \underline{b}'\|$$

But remember, the absolute condition number tells us nothing about the number of correct digits in the answer.

Need the relative condition number:

$$\frac{\|\underline{x}-\underline{x}'\|}{\|\underline{x}\|} \leq \|A^{-1}\| \frac{\|\underline{b}-\underline{b}'\|}{\|\underline{x}\|}$$

$$= \|A^{-1}\| \frac{\|\underline{b}-\underline{b}'\|}{\|\underline{b}\|} \underbrace{\frac{\|\underline{b}\|}{\|\underline{x}\|}}_{= \frac{\|A\underline{x}\|}{\|\underline{x}\|} \leq \|A\|}$$

$$\Rightarrow \leq \underbrace{\|A\| \, \|A^{-1}\|}_{\text{relative condition number.}} \frac{\|\underline{b}-\underline{b}'\|}{\|\underline{b}\|}$$

What is this quantity? $\|A\| \cdot \|A^{-1}\|$

It is defined as the condition number of A:

$$K(A) = \|A\| \, \|A^{-1}\|$$

Ex (In the 2-norm).

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$$

What is $\|A\|$, $\|A^{-1}\|$?

largest eigenvalue of $A^tA = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 10 & 8 \\ 8 & 8 \end{pmatrix}$

$A\underline{v} = \lambda \underline{v} \Rightarrow$ $A - \lambda I$ must be singular:

$$\begin{vmatrix} 10-\lambda & 8 \\ 8 & 8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 18\lambda + 80 - 64 = 0$$

$$\lambda^2 - 18\lambda + 16 = 0 \qquad \Rightarrow \lambda: \frac{18 \pm \sqrt{324 - 4 \cdot 16}}{2}$$

$$= \frac{9 \pm \sqrt{260}}{2} : 9 \pm \sqrt{65}$$

$$\Rightarrow \|A\| = 9 + \sqrt{65}$$

$$(A^tA^{-1}) = \frac{1}{10\cdot 8 - 8\cdot 8}\begin{pmatrix} 8 & -8 \\ -8 & 10 \end{pmatrix}$$

$$= \frac{1}{80-64}\begin{pmatrix} 8 & -8 \\ -8 & 10 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 5/8 \end{pmatrix}$$

$$\Rightarrow \|A^{-1}\| = \frac{9+\sqrt{17}}{2}$$

$$\Rightarrow \begin{vmatrix} \frac{1}{2}-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{8}-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \frac{9}{8}\lambda + \frac{5}{16} - \frac{1}{4} = 0$$

$$\lambda^2 - \frac{9}{8}\lambda + \frac{1}{16} = 0$$

$$16\lambda^2 - 9\lambda + 1 = 0$$

$$\lambda: \frac{9 \pm \sqrt{81 - 4 \cdot 16}}{2}$$

$$= \frac{9}{2} \pm \frac{\sqrt{81-64}}{2} = \frac{9 \pm \sqrt{17}}{2}$$

$\|A\|_2$ computation.

what is the 2-norm of a matrix?

$$\|A\|_2^2 = \max_{\|\underline{u}\|=1} \|A\underline{u}\|_2^2$$

$$= \max_{\|\underline{u}\|=1} (A\underline{u}, A\underline{u})$$

$$= \max (\underline{u}, A^TA\underline{u})$$

$B = A^TA$ is symmetric,

$\Rightarrow \lambda$ are real and nonnegative.

$\Rightarrow \vec{V}_j$ are orthogonal

$$\Rightarrow \max (\underline{u}, VDV^T\underline{u})$$

orthogonal and invertible, just a change of basis, $\|V^T\underline{u}\| = \|\underline{u}\|$

$$= \max_{\|\underline{u}\|=1} (V^T\underline{u}, DV^T\underline{u})$$

$$= \max_{\|\underline{v}\|=1} (\underline{v}, D\underline{v}) = \max_j \lambda_j$$

$\Rightarrow$ Therefore, $\|A\|_2 = \max_j \sqrt{\lambda_j}$ where λ_j are eigenvalues of A^TA.

What else is related to the eigenvalues of A^tA? Recall the singular-value decomposition:

For <u>any</u> matrix A (square or not, invertible or not)

$A = U S V^t$ — U, V^t orthogonal — $U^t U = I$; S diagonal, $\geq$ entries

If A <u>is</u> invertible, then $A^tA = (VSU^t)(USV^t)$

$= (VS^2V^t)$

~~Computing the SVD: (Numerically)~~ We will return to computing the SVD numerically...

Interpretation of the ℓ^2-condition number: <u>ratio of stretching</u> to <u>shrinking</u>.

$\|A\|$ = max eigenvalue for A^tA → corresponds to eigenvector of A

$\|A^{-1}\|$ = min " " " " " " "

$(A^tA)^{-1} = V S^{-2} V^t$ If $S^2 = \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{pmatrix}$ $\sigma_1^2 > \cdots > \sigma_n^2$,

then $\|A\| = \sigma_1$

$\|A^{-1}\| = \frac{1}{\sigma_n}$

$S^{-2} = \begin{pmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{pmatrix}$ ($\frac{1}{\sigma_n^2}$ largest)

$\Rightarrow \boxed{K(A) = \frac{\sigma_1}{\sigma_n}}$