Lecture 9 Numerical Analysis 2/20/18

Last Time

- Matrix and vector norms.

$$\|\underline{u}\|_\infty = \max_i |u_i|$$

$$\|\underline{u}\|_1 = \sum_i |u_i|$$

$$\|\underline{u}\|_2 = \sqrt{\sum_i |u_i|^2}$$

- Induced matrix norm:

$$\|A\| = \max_{\|\underline{u}\|=1} \|A\underline{u}\|.$$

- We are studying norms in order to analyze the error ~~induced~~ incurred via solving problems in floating point arithmetic with roundoff error.

Today

- conditioning of problems
- condition number of a matrix.

Condition number of a problem

$\Rightarrow$ The sensitivity of the "problem" at the solution.

Ex: For a function $y = f(x)$, how sensitive is y to x? ~~→ one estimate is the rde.~~ (The "problem" is computing $f(x)$.)

① In an absolute sense:

$$|y - y'| \approx \underbrace{C(x)}_{\text{absolut condition number}} |x - x'|$$

$$\Rightarrow C(x) \approx \frac{|y - y'|}{|x - x'|} = \frac{|f(x) - f(x')|}{|x - x'|}$$

for x' close to x, $C(x) \approx |f'(x)|$.

② In a relative sense:

$\frac{|x - x'|}{|x|}$ = relative error = # digits that are correct

$$\frac{|y' - y|}{|y|} \approx K(x) \left| \frac{x' - x}{x} \right|$$

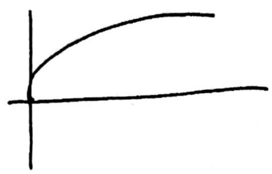
$$\Rightarrow K(x) \approx \left| \frac{y' - y}{y} \right| \left| \frac{x}{x' - x} \right| = \left| \frac{f(x') - f(x)}{x' - x} \right| \left| \frac{x}{y} \right| = \left| \frac{x f'(x)}{f(x)} \right|$$

Ex: Let $y = x^{\frac{1}{3}} = f(x)$

$$C(x) = \frac{1}{3} x^{-2/3}$$

$$K(x) = \frac{x f'(x)}{f(x)} = \frac{x \cdot \frac{1}{3} \cdot x^{-2/3}}{x^{1/3}} = \frac{\frac{1}{3} x^{1/3}}{x^{1/3}} = \frac{1}{3}$$

$C(x) = \frac{1}{3} \frac{1}{x^{2/3}} < \infty$ for x away from 0



$K(x)$ is small <u>everywhere</u>

~~when $x \to x+\epsilon$, $f(x) \to f(x) + C(x)\epsilon = f(x) + \frac{1}{3}\frac{1}{x^{2/3}}\epsilon$~~

$$f(x+\epsilon) \approx f(x) + f'(x)\epsilon \approx f(x) + \frac{1}{3}x^{-2/3}\epsilon$$

$$\Rightarrow |f(x+\epsilon) - f(x)| \approx \underbrace{\frac{1}{3x^{2/3}}}_{\text{large for } x \text{ near } 0}\epsilon$$

~~If $\epsilon \sim 10^{-16}$, then $f(x) \approx f(x) + \frac{1}{3}10^{5}$!!~~

~~Relative (Given $y = f(x)$, how accurate can we calculate x?~~

~~We need an idea of~~

Stability of an algorithm

Stable: gives "correct" level of accurate ~~relative~~ for a problem with a given condition number

unstable: For well conditioned problems, gives inaccurate solution.

=> Rounding error analysis.

Ex: $\underbrace{fl(x+y)}_{\text{floating point result}} = \text{round}(x) \oplus \text{round}(y)$

$\uparrow$ ~~comp~~ fp-addition

$$= \left(x(1+\delta_1) + y(1+\delta_2)\right)(1+\delta_3)$$

each $\delta_i < \epsilon$ ← machine precision

Fwd error analysis What is the accuracy of the forward operation? How does error propagate?

↑ roundoff error

$$fl(x+y) = ~~x~~\ x+y + x(\delta_1+\delta_3+\delta_1\delta_3) + y(\delta_2+\delta_3+\delta_2\delta_3)$$

Absolute error: $|fl(x+y) - (x+y)| \le (|x|+|y|)(2\epsilon+\epsilon^2)$

Relative Error

$$\left| \frac{fl(x+y) - (x+y)}{x+y} \right| \leq \underbrace{\frac{(|x|+|y|)(2\epsilon+\epsilon^2)}{|x+y|}}$$

if $y \approx -x$ the relative error is large!

Backward Error analysis:

Ex: Solve $f(x) = b$.

Is the x I find the true x for some b' which is close to b?

Ex: $$fl(x+y) = \underbrace{x(1+\delta_1)(1+\delta_3) + y(1+\delta_2)(1+\delta_3)}$$

the true sum of two numbers which are very close to x, y.

Ex: Finite difference:

$$fl\left(\frac{f(x+h) - f(x)}{h}\right) \approx \frac{f(x+h)(1+\delta_1) - f(x)(1+\delta_2)}{h}$$

$$= \frac{f(x+h) - f(x)}{h} + \frac{f(x+h)\delta_1 - f(x)\delta_2}{h}$$

$$\Rightarrow \left| fl\left(\frac{f(x+h) - f(x)}{h}\right) - \frac{f(x+h) - f(x)}{h} \right| \lesssim \frac{2 f(x) \epsilon}{h}$$

In single precision, $\epsilon \sim 10^{-7}$.
If $h \sim 10^{-8}$,
$\Rightarrow$ error ≈ 20!

What does this have to do with the matrix condition number?

By our definition earlier: sensitivity of the solution to the input

$$\begin{array}{cc} \text{solution} & \text{input} \\ \uparrow & \uparrow \\ \underline{x} & \underline{b} \end{array}$$

Let $\|\underline{b} - \hat{\underline{b}}'\|$ be small, and $\underline{x} = A^{-1}\underline{b}$, $\underline{x}' = A^{-1}\underline{b}'$.

Then $\|\underline{x} - \underline{x}'\| = \|A^{-1}\underline{b} - A^{-1}\underline{b}'\|$

$$\le \underbrace{\|A^{-1}\|}_{\text{absolute condition number}} \|\hat{\underline{b}} - \underline{b}'\|$$

But remember, the absolute condition number tells us nothing about the number of correct digits in the answer.

Need the relative condition number:

$$\frac{\|\underline{x} - \underline{x}'\|}{\|\underline{x}\|} \le \|A^{-1}\| \frac{\|\underline{b} - \underline{b}'\|}{\|\underline{x}\|}$$

$$= \|A^{-1}\| \frac{\|\underline{b} - \underline{b}'\|}{\|\underline{b}\|} \underbrace{\frac{\|\underline{b}\|}{\|\underline{x}\|}}_{= \frac{\|A\underline{x}\|}{\|\underline{x}\|} \le \|A\|}$$

$$\Rightarrow \le \underbrace{\|A\| \, \|A^{-1}\|}_{\text{relative condition number.}} \frac{\|\underline{b} - \underline{b}'\|}{\|\underline{b}\|}$$