# Lecture 8 — Numerical Analysis — 2/15/18

Last time:

- Gaussian elimination
- LU Factorization — $O(n^3)$ vs $O(n^2)$ back solve
- Operation counts
- Pivoting <u>not</u> necessary for SPD matrices.

One last note: HPC

- on modern machines, memory access is expensive
- BLAS, LAPACK
- Blocking $\rightarrow$ $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

Warning:

1. Factoring $A = PLU$ is preferable and cheaper than $A \rightarrow A^{-1}$
2. <u>Never use Cramer's rule</u>
   - Unstable
   - computationally expensive with determinants

<u>Next topic:</u> Norms & Conditioning

We will now discuss the analogue for linear systems: When solving $A\vec{x} = \vec{b}$, what is the sensitivity of $\vec{x} = A^{-1}\vec{b}$ on $\vec{b}$?

$\Rightarrow$ say something about "norms" $\|\vec{x}\| \leq \|A^{-1}\| \|b\|$ ?

## Norms for vectors:

Def: $\|\cdot\|$ is a norm if

(1) $\|\vec{u}\| \geq 0$, $\|\vec{u}\| = 0$ iff $\vec{u} = \vec{0}$

(2) $\|\alpha \underline{u}\| = |\alpha| \, \|\underline{u}\|$

(3) $\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$ (triangle inequality)

Think about norms as lengths $\Rightarrow$ $\|\underline{u}\| = \sqrt{\underline{u} \cdot \underline{u}}$

$= \sqrt{\sum u_i^2}$ ($\ell^2$ norm)

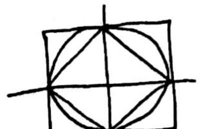
## Alternative norms

$\ell^\infty$-norm: $\|\underline{u}\|_\infty = \max |u_i|$

$\ell^1$-norm: $\|\underline{u}\|_1 = \sum |u_i|$

$\ell^p$-norm: $\|\underline{u}\|_p = \left( \sum |u_i|^p \right)^{1/p}$

Ex: "unit circles in $1, \infty, 2$ norms"



Only the 2-norm comes from an inner product (dot product)

That was for vectors — how about for matrices?

Def: Matrix norm $\|A\|$: ( same rules ! )

(1) $\|A\| \geq 0$

(2) $\|\alpha A\| = |\alpha| \|A\|$

(3) $\|A+B\| \leq \|A\| + \|B\|$ ( norms are a "thing" of mathematics )

[ (4) $\|AB\| \leq \|A\| \cdot \|B\|$ (sometimes) ]

If $\|\underline{u}\|$ is any norm on a vector, then the induced matrix norm is:

$$\|A\| = \max_{\|\underline{u}\|=1} \|A\underline{u}\| = \max \frac{\|A\underline{u}\|}{\|\underline{u}\|}$$

$\Rightarrow$ For any $\|\underline{u}\|$, $\|A\underline{u}\| \leq \|A\| \|\underline{u}\|$.

Thm: If $\|\underline{u}\| = \|\underline{u}\|_1 = \sum |u_i|$ then the induced norm ~~on A~~ on matrix is:

$$\|A\| = \max_{j=1,n} \sum_{i=1}^{m} |a_{ij}|$$

( max over columns sum over rows )

Pf: $\|A\underline{u}\| = \|\sum \underline{a}_j u_j\| \leq \sum \|\underline{a}_j\| |u_j| \leq \max \|\underline{a}_j\| \sum |u_j|$

$= \max \|\underline{a}_j\| \|\underline{u}\|$

But $\underline{u} = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \Rightarrow \max \|\overset{A\underline{u}}{\underline{a}_j}\| \geq \|a_1\|$ $\square$

Alternatively:

**Thm** $\|A\|_\infty = \max_{i=1,m} \sum_{j=1}^{n} |a_{ij}|$

**pf:** $\|A\underline{u}\|_\infty = \max_{i=1,m} |(A\underline{u})_i| = \max_i \left|\sum_j a_{ij} u_j\right|$

$\leq \max_i \sum_j |a_{ij}|\,|u_j|$

$\leq \max_i \left(\max_j u_j\right) \sum_j |a_{ij}|$

$= \|\underline{u}\|_\infty \max_i \sum_j |a_{ij}|$

$\Leftarrow$ set $\underline{u} = \underline{1}$ to pick out maximal row.

Lastly, and most commonly used:

**Thm** $\|A\|_2 = \sqrt{\max_j |\lambda_j^2|}$ of $A^tA$

**Pf** $\|A\underline{u}\|_2^2 = (A\underline{u}, A\underline{u}) = (\underline{u}, A^tA\underline{u})$

$A^tA$ is SPD, and has real eigenvalues

$\Rightarrow$ diagonalizable $\Rightarrow$ $(\underline{u}, PDP^t\underline{u})$

$= (P^t\underline{u}, DP^t\underline{u})$

$\max_{\underline{u}} (P^t\underline{u}, DP^t\underline{u}) = \max_{\underline{v}=1} (\underline{v}, D\underline{v}) =$ largest eigenvalue.

$\Rightarrow$ Rayleigh quotient: $\dfrac{\underline{x}^tA\underline{x}}{\underline{x}^t\underline{x}}$ (also largest singular value.)