Last time:

- Fixed point iteration

- (Keep in mind that all our root finding algorithms are just fixed point methods. Ex: Newton:

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x)} \]

\[ \iff \quad x = x - \frac{f(x)}{f'(x)} \]

\[ \implies \text{trivially true if } f(x) = 0, \]

- Statement of the Contraction Mapping Theorem.
Recall:

**Definition (Contraction)** Let \( g \) be continuous on \([a,b]\). The function \( g \) is a contraction on \([a,b]\) if there exists a number \( L \) with \( 0 < L < 1 \) such that

\[
|g(x) - g(y)| < L |x - y| \quad \text{for all } x, y \in [a,b].
\]

This means that \( g \) maps points to values which are closer together.

If \( L \) is allowed to be any positive number, this is known as a **Lipschitz condition**.
Then Contraction Mapping Theorem

Let \( g \) be continuous on \([a,b]\), \( g(x) \in [a,b]\),
and \( g \) is a contraction on \([a,b]\).

Then \( g \) has a unique fixed point
\( s = g(s) \). Furthermore, \( x_{k+1} = g(x_k) \) converges
to \( s \) for any \( x_0 \in [a,b] \).

**Proof:** A fixed point \( s \) exists due
to Brouwer's Fixed Point Thm. Now,
suppose that \( s \) is not unique. Then there
exists another fixed point, \( s' \), such
that
\[
|s - s'| = |g(s) - g(s')| \leq L |s - s'|,
\]
so
\[
(1 - L) |s - s'| > 0 \quad \Rightarrow \quad (1 - L) > 0.
\]
Contradiction.
Now, we will show that \( \{x_k\} \) converges for any initial \( x_0 \in [a, b] \).

Since \( g \) is a contraction,
\[
|x_k - s| = |g(x_{k-1}) - g(s)| \leq L |x_{k-1} - s|
\]

\[
\Rightarrow |x_k - s| \leq L^k |x_0 - s|
\]

and given \( 0 < L < 1 \), \( \lim_{k \to \infty} L^k = 0 \),

and therefore \( \lim_{k \to \infty} |x_k - s| = 0 \).

\[\text{Ex: Look at } f(x) = e^x - 2x - 1 \text{ on } [1, 2].\]
\( f(x) = 0 \) has a solution, \( \xi \). This

\[\text{can be re-written as a fixed point problem:}\]
\[ e^x - 2x - 1 = 0 \]
\[ \Rightarrow e^x = 2x + 1 \]
\[ \log e^x = \log (2x+1) \]
\[ x = \log (2x+1) \]

\text{Denote this function } g. \\

\text{g is continuous on } [1,2], \text{ and differentiable.} \\

By the MVT, for some } \eta \in [1,2] \text{,}
\[ |g(x) - g(y)| = |g'(\eta)(x-y)| \]
\[ = |g'(\eta)| |x-y| \]
\[ \leq \frac{1}{2x+1} \cdot \frac{2}{3} \in \left[ \frac{2}{3}, \frac{2}{3} \right] \]
\[ \leq \frac{2}{3} |x-y| \text{ So } g \text{ is a contraction.} \]

Therefore, by the Contraction Mapping Theorem, } x_{k+1} = g(x_k) \text{ converges for any } x_0 \in [1,2] \text{ to } \xi, \text{ the root of } f.
How many iterations do we need to do to guarantee that $|x_n - s| \leq \epsilon$?

Well, from CMT, we have:

$$|x_n - s| \leq L^n |x_0 - s|$$

$$\leq L^n |b - a|$$

trivially.

If $L^n |b - a| \leq \epsilon$, then

$$L^n \log L \leq \log \frac{\epsilon}{|b - a|}$$

$$\Rightarrow L^n \geq \frac{1}{\log L} \log \frac{\epsilon}{|b - a|}$$

note since $\log L$ is negative.

Now a few notes on convergence:

**Stable fixed point**

$s$ is stable if $x_n \to s$ for every $x_0$ in some sufficiently small neighborhood of $s$. 

Unstable fixed point

$s$ is unstable if the only initial condition that yields a convergent sequence is $x_0 = s$ (i.e., $x_k$ diverges for every $x_0$ in a neighborhood of $s$).

If $s$ is a stable fixed point, at what rate does the sequence converge?

Let $x_k$ be the ratio of successive errors:

$$\lim_{k \to \infty} \frac{|x_{k+1} - s|}{|x_k - s|} = |g'(s)|$$

So the derivative at $s$ dictates how fast the sequence converges.

Note: $|g'(s)| < 1$, otherwise $|x_{k+1} - s| > |x_k - s|$ and the seq. diverges.
Definitions of rates:

Let the error be

$$|x_k - s| = e_k,$$

and set

$$\lim_{k \to \infty} \frac{e_{k+1}}{e_k} = \mu.$$

If $\mu \in (0, 1)$ then $\{x_k\}$ converges linearly.

Set $\rho = -\log_{10} \mu$.

$\rho$ is the asymptotic rate of convergence.

Example:

$$x_k = 1 + \frac{1}{10^k}$$

$$\lim_{k \to \infty} \frac{|1 + \frac{1}{10^k} - 1|}{|1 + \frac{1}{10^k} - 1|} = \lim_{k \to \infty} \frac{1}{10^k} \times \frac{1}{10^k} = \frac{1}{10}$$

$$\rho = -\log_{10} \frac{1}{10} = \log_{10} 10 = 1$$
What does $\rho$ measure: the number of correct decimal digits gained in one iteration.

For fixed point iterations, since

$$\lim_{n \to \infty} \frac{e_{n+1}}{e_n} = |g'(s)|, \quad M = |g'(s)|$$

and

$$\rho = -\log_{10} |g'(s)|$$

This means that the flatter the function $g$ is near the fixed point, the faster $x_{n+1} = g(x_n)$ converges.

Example

![Graphs showing convergence](graphs.png)