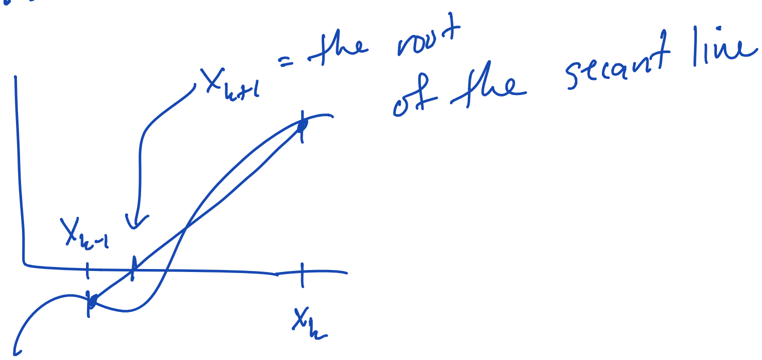
Lecture 2 : Numerical Analysis 1/25/18

Last time:

- Overview
- Bisection Method for solving $\underbrace{f(x)=0}_{\text{Nonlinear function}}$
- Secant Method



the secant line $s(x) = \dfrac{f_k - f_{k-1}}{x_k - x_{k-1}}(x - x_k) + f_k$

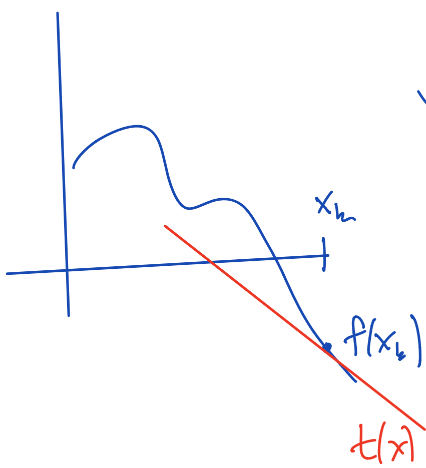
Its zero solves $s(x) = 0$

$$\Rightarrow \quad x = x_k - f_k \frac{x_k - x_{k-1}}{f_k - f_{k-1}}$$

- Convergence rate to follow...
- This method only used function values to find the next approx. root.

What if we now want to incorporate actual derivative information? $\Rightarrow f'(x_k)$?

Using only two pieces of information, $f(x_k)$, $f'(x_k)$ (the same as before) we can Taylor expand f about x_k as a <u>linear</u> function:



$$\underbrace{f(x) \approx f(x_k) + f'(x_k)(x - x_k)}$$

this is the equation of the tangent line to f at $x_k \rightarrow t(x)$

We can then solve for x s.t. $t(x) = 0$ and use this as an approximate root of f, denoting it by x_{k+1}.

The root of t satisfies:

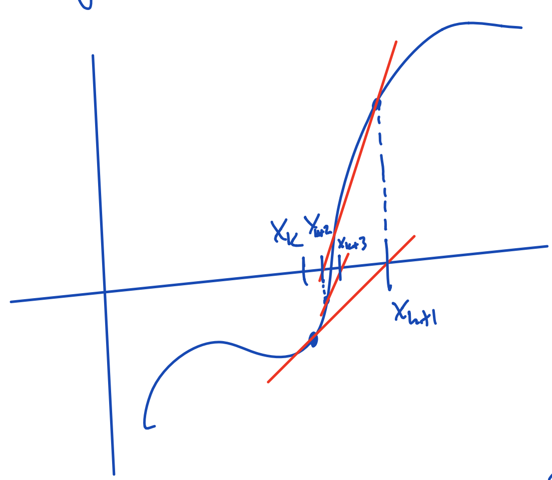
$$t(x) = 0 = f(x_k) + f'(x_k)(x - x_k)$$

$$\Rightarrow \quad x = x_k - \frac{f(x_k)}{f'(x_k)}$$

This leads to the iteration:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$ Newton's Method

Graphically,



In this (not-exact) example, after some erratic behavior, the sequence starts converging.

How fast does this converge?

Assume that the sequence does converge for a moment. Expand further in a Taylor series:

$$f(x) = f(x_k) + f'(x_k)(x-x_k) + \frac{f''(x_k)}{2}(x-x_k)^2 + \ldots$$

Using Taylor's Thm, we can truncate this series and evaluate f'' at a new point $\eta_k \in [x_k, x]$:

$$f(x) = f(x_k) + f'(x_k)(x-x_k) + \frac{f''(\eta_k)}{2}(x-x_k)^2 \quad (*)$$

Denote the true root of f by ξ, and recall Newton's Method:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

subtract ξ from both sides: to find the error in x_{k+1}

$$\Rightarrow \quad \xi - x_{k+1} = \xi - x_k + \frac{f(x_k)}{f'(x_k)}$$

But from (*) we have that

$$\frac{f(x_k)}{f'(x_k)} = -(\xi - x_k) - \frac{f''(\eta_k)(\xi - x_k)^2}{2f'(x_k)}$$

Inserting this into Newton's Method we have:

$$\xi - x_{k+1} = \xi - x_k - (\xi - x_k) - \frac{f''(\eta_k)(\xi - x_k)^2}{2f'(x_k)}$$

$$= -\frac{f''(\eta_k)(\xi - x_k)^2}{2f'(x_k)}$$

So the absolute error $|\xi - x_{k+1}| = \left|\frac{f''(x_k)}{2f'(x_k)}\right| |\xi - x_k|^2.$

This is known as quadratic convergence, and it is extremely fast.

If $|\xi - x_k| \sim 10^{-3}$, $|\xi - x_{k+1}| \sim 10^{-6}$

$|\xi - x_k| \sim 10^{-6}$, $|\xi - x_{k+1}| \sim 10^{-12}$

$\vdots$

The computer bottoms out at $\sim 10^{-16}$!

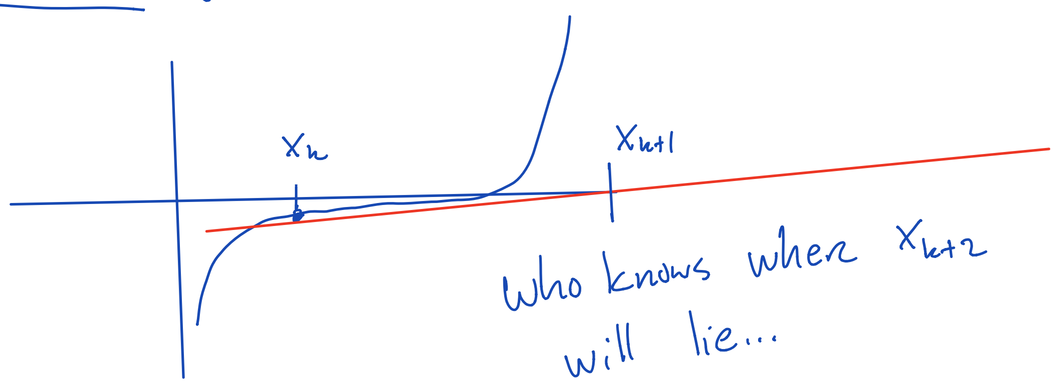
What were our basic assumptions here?

1 – that the iteration actually converged (not so straightforward)

2 – $f' \neq 0$ (since we were dividing by it)

3 – f'' is bounded in the neighborhood of ξ, otherwise the estimates don't make any sense.

Example (of bad Newton behavior)



who knows where x_{n+2} will lie...

We can state these assumptions and result precisely now:

[Thm 1.8] Suppose that f is twice continuously differentiable (f, f', f'' are cont.) on the interval $[\xi-\delta, \xi+\delta]$, $\delta > 0$, and that $f(\xi) = 0$ and $f''(\xi) \neq 0$. Also assume that there exists $A > 0$ s.t.

$$\left| \frac{f''(x)}{f'(y)} \right| \leq A \quad \text{for all } x, y \in [\xi-\delta, \xi+\delta].$$

If $|\xi - x_0| \leq h$, where $h \leq \delta, 1/A$, then the sequence $\{x_k\}$ (with starting guess x_0) defined by Newton's method converges quadratically to ξ.

<u>Proof</u>: Let $|\xi - x_k| \leq h \leq \min(\delta, \frac{1}{A})$ so that $x_k \in [\xi-\delta, \xi+\delta] = I_\delta$. Then (as before) by Taylor's Thm,

$$0 = f(\xi) = f(x_k) + f'(x_k)(\xi - x_k) + \frac{f''(\underbrace{\eta_k}_{\text{in } [\xi, x_k]})}{2}(\xi - x_k)^2.$$

$\rightarrow$ η_k is also in I_δ.

This gives us our above result of

$$\xi - x_{k+1} = \frac{-f''(\eta_k)}{2f'(x_k)} (\xi - x_k)^2$$

And therefore

$$|\xi - x_{k+1}| \leq \left| \frac{f''(\eta_k)}{2f'(x_k)} \right| |\xi - x_k|^2$$

$$\leq \frac{1}{2} A \frac{1}{A} |\xi - x_k| = \frac{1}{2} |\xi - x_k|$$

Since $|\xi - x_0| \leq h$, we have that $|\xi - x_k| \leq \frac{1}{2^k} h$,

and therefore $x_k \to \xi$ as $k \to \infty$.

Furthermore,

$$\lim_{k \to \infty} \frac{|\xi - x_{k+1}|}{|\xi - x_k|} = \lim_{k \to \infty} \left| \frac{f''(\eta_k)}{2f'(x_k)} \right| = \underbrace{\left| \frac{f''(\xi)}{2f'(\xi)} \right|}_{\text{A constant.}}$$

This is the definition of quadratic convergence.