Student Notes Nov 16

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1 Lecture Video

1.1 Conditional Distribution: Discrete Case

Conditional Distributions

Goal: Given X,Y random variables with joint pdf f(x,y) and marginal pdfs: f_x, f_y, what is the pdf of the random variables :

$$z = X|Y = y$$

Recall for events:

$$P[E|F] = P[EF]/P[F]$$

Define the **conditional probability mass function** so that:

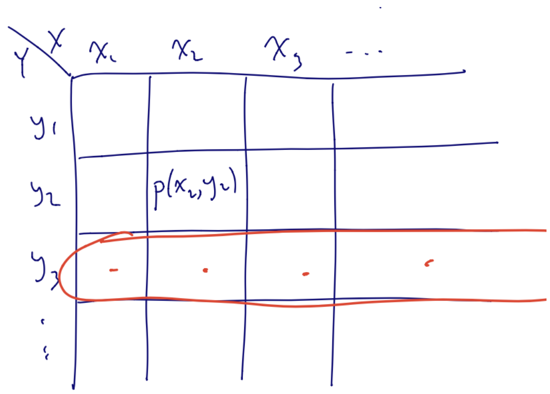
$$P_{X|Y}(x|y) = P[X = x|Y = y] = \frac{P[X = x, Y = y]}{P[Y = y]} = \frac{p(x,y)}{P_Y(y)}$$

$p(x, y)$: joint mass function
$P_Y(y)$: marginal mass function (Assuming $P_Y(y) > 0$)

Conditional distribution function is then:

$$F_{X|Y}(x|y) = P[X \leq x|Y \leq y] = \sum_{a \leq x} P[X = a|Y = y]$$

Pictorially, imagine there is a grid of probabilities:



Fix $Y = y_3$, scale this row by $\frac{1}{P_Y(y_3)}$ so that $\sum P_{X|Y}(x_i|y_3)$

If X and Y are **independent random variables**, then:

$$p(x, y) = P_X(x)P_Y(y)$$

$$\rightarrow P_{X|Y}(x|y) = \frac{P(x,y)}{P_Y(y)} = \frac{P_X(x)P_Y(y)}{P_Y(y)} = P_X(x)$$

1.2 Conditional Distribution: Continuous Case

If X and Y have a joint probability density function $f(x, y)$, then the conditional probability density function of X given that Y = y is defined, for all values of y such that $f_Y(y) > 0$,by:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

To motivate this definition, multiply the left-hand side by dx and the right-hand side by (dx dy)/dy to obtain

$$f_{X|Y}(x|y) = \frac{f(x,y)dxdy}{f_Y(y)dy}$$

$$\cong \frac{P\{x <= X <= x + dx, y <= Y <= y + dy\}}{P\{y <= Y <= y + dy\}}$$

$$= P\{x <= X <= x + dx, y <= Y <= y + dy\}$$

The use of conditional densities allows us to define conditional probabilities of events associated with one random variable when we are given the value of a second random variable. That is, if X and Y are jointly continuous, then, for any set A,

$$f_{X|Y}(x|y) = \frac{f(x,y)dxdy}{f_Y(y)dy}$$

The use of conditional densities allows us to define conditional probabilities of events associated with one random variable when we are given the value of a second random variable. That is, if X and Y are jointly continuous, then, for any set A,

$$P(X \in A|Y = y) = \int_A f_{X|Y}(x|y)dx$$

Likewise,

$$P(X \in A|Y \in B) = \int_A \int_B f_{X|Y}(x|y)dxdy$$

For A in range (-$\infty, x), the distribution function is:$

$$f_{X|Y}(x|y) = P[X <= x|Y = y]$$
$$= \int_{-\infty}^{x} f_{X|Y}(u|y)du$$

If X and Y are independent, then:

$$f_{X|Y}(x|y) = f_X(x)$$

2 In Class Examples

2.1 Discrete Conditional Distribution

$f(x,y)=\begin{cases} \frac{e^{\frac{-x}{y}} e^{-y}}{y}, & \text{if } x, y > 1 \\ 0, & \text{otherwise} \end{cases}$

What is $P[X > 1|Y = y]$?

$$f_{X|Y}(x|y) = \frac{f(xy)}{f_Y(y)}$$

$$
\begin{aligned}
f_Y(y) &= \int_0^\infty f(x,y)dx = \int_0^\infty \frac{e^{\frac{-x}{y}} e^{-y}}{y} dx \\
&= \frac{e^{-y}}{y} \int_0^\infty e^{\frac{-x}{y}} dx \\
&= (\frac{e^{-y}}{y}) - ye^{\frac{x}{y}} \Big|_0^\infty \\
&= \frac{e^{-y}}{y} (0 + ye^0 \\
&= e^{-y}
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f(xy)}{f_Y(y)} \\
&= \frac{e^{\frac{-x}{y}} e^{-y}/y}{e^{-y}} \\
&= \frac{e^{\frac{-x}{y}}}{y}
\end{aligned} \tag{2}
$$

Then,

$$
\begin{aligned}
P[X > 1|Y = y] &= \int_1^\infty f_{X|Y}(x|y)dx = \int_1^\infty \frac{e^{\frac{-x}{y}}}{y} dx \\
&= (\frac{1}{y}) - ye^{\frac{-x}{y}} \Big|_1^\infty \\
&= (\frac{1}{y})(0 + ye^{\frac{-1}{y}}) \\
&= e^{\frac{-1}{y}}
\end{aligned} \tag{3}
$$

Example 4B (pg. 264):

If X and Y are independent Poisson variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X given that X+Y = n.

We calculate the conditional probability mass function of X given that X+Y=n as follows:

$$
PX = k|X + Y = n = \frac{P[X = k, X + Y = k]}{P[X + Y = n]} = \frac{P[X = k, Y = n - k]}{P[X + Y = n]} = \frac{P[X = k]P[Y = n - k]}{P[X + Y = n]}
$$

The last equality follows from the assumed independence of X and Y. Recalling that X+Y has a Poisson distribution with parameter $\lambda_1 + \lambda_2$, we see that it equals:

$$
\begin{aligned}
P[X = k|X + Y = n] &= (\frac{e^{\lambda_1}\lambda_1^k}{k!})(\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!})[\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}]^{-1} \\
&= (\frac{n!}{(n-k)!k!})(\frac{\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n}) \\
&= \binom{n}{k}(\frac{\lambda_1}{\lambda_l+\lambda_2})^k(\frac{\lambda_2}{\lambda_1+\lambda_2})^{n-k}
\end{aligned}
\tag{4}
$$

The conditional distribution of X given that X+Y=n is the binomial distribution with parameters n and $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

2.2 Continuous Conditional Distribution

5a (page 266):

The joint density of X and Y is given by:

$$
f(x,y) = \begin{cases} 12x(2-x-y), 0 < x < 1, 0 < y < 1 \\ 0, otherwise \end{cases} \tag{5}
$$

Question: Compute the conditional density of X given that Y=y, given that $0 < y < 1$.

Solution: For $0 < x < 1, 0 < y < 1$, we have:

$$
f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}
$$

$$
= \frac{f(x,y)}{\int_{-\infty}^{+\infty} f(x,y)dx}
$$

$$
= \frac{x(2-x-y)}{\int_0^1 x(2-x-y)dx}
$$

$$
= \frac{x(2-x-y)}{2/3 - y/2}
$$

$$
= \frac{6x(2-x-y)}{4-3y}
$$

5b (page 267): Suppose that the joint density of X and Y is given by:

$$
f(x,y) = \begin{cases} \frac{e^{-x/y}e^{-y}}{y}, 0 < x < \infty, 0 < y < \infty \\ 0, otherwise \end{cases} \tag{6}
$$

Find $P[X > 1|Y = y]$.
Solution:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{e^{-x/y}e^{-y}/y}{e^{-y}\int_0^\infty (1/y)e^{-x/y}dx}$$

$$= \frac{e^{-x/y}}{y}$$

Hence,

$$P(X > 1|Y = y) = \int_1^\infty e^{-x/y}/y dx$$

$$= e^{-1/y}$$

When X and Y are independent,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x)$$

Suppose that X is a continuous random variable having probability density function f and N is a discrete random variable, and consider the conditional distribution of X given that N = n. Then:

$$\frac{P(x < X < x + dx|N = n)}{dx} = \frac{P(N = n|x < X < x + dx)P(x < X < x + dx)}{P(N = n)dx}$$

Letting dx approach 0, then:

$$\lim_{dx \to 0} \frac{P(x < X < x + dx|N = n)}{dx} = \frac{P(N = n|X = x)f(x)}{N = n}$$

Therefore,

$$f_{X|N}(x|n) = \frac{P(N = n|X = x)f(x)}{N = n}$$