1 Lecture Notes

1.1 Independent Random Variables:

X,Y are independent random variables if for any two sets of real numbers A,B,

\[ P\{X \in A, Y \in B\} = P\{X \in A\} \cdot P\{Y \in B\} \]

The joint CDF is equal to

\[ F(a,b) = F_X(a)F_Y(b) \]

The joint probability density, \( f(x,y) \), is equal to

\[ f(x)f_y(y) \]

If the two random variables are not independent and don’t satisfy the above criteria, then we say that they are dependent.

Here is an example:

\[ f(x,y) = 6e^{-2x} \cdot e^{-3y}, x,y \geq 0 \]

\[ f(x,y) = f_x.f_y = 2e^{-2x} \cdot 3e^{-3y}, \text{ where } x \text{ and } y \text{ are independent because we were able to split them up into two different functions.} \]

Here is another function, albeit dependent. \( f(x,y) = 24xy, x \epsilon (0,1), y\epsilon(0,1), x+y \leq 1 \) The function is 0 otherwise. This function will lie in the region bounded between the x and y axes and x+y=1. This function can’t be split into separate functions of x and y because the domain of dependence can’t be split into functions of x and y. The reason for this is because the indicator function of \( f(x,y) \) is a function of \( x+y \), not \( xy \).
1.2 Sums of Independent Random Variables:

If X and Y are independent, let Z=X+Y. We need to calculate the PDF and CDF of Z. Its CDF is \( F_Z(z) = P[Z \leq z] = P[X + Y \leq z] = \int_D f(x, y) \, dx \, dy \) where \( D: x + y \leq z \). Since X and Y are independent, we can factor. Our last equation is equal to \( \int_D f_X(x) f_Y(y) \, dx \, dy \). Our domain consists of all the points to the "left" of the line x+y=z. So our equation can be rewritten as:

\[
\sum_{k=0}^{n} f_X(z-y) f_Y(y) \, dy = \int_0^\infty f_X(z-y) f_Y(y) \, dy
\]

Another important concept are the Independent Identically Distributed (IID) Random Variables. Consider two IID U(0,1) RVs X,Y. \( Z=X+Y \).

Propositions for Continuous RVs:

Proposition 1: If \( f_X(z-y) \) is non-zero when \( \int_0^1 1 \, dy = z \) then \( f_X(z-y) \) is non-zero when \( \int_0^1 1 \, dy = z \) and is 0 otherwise. When you add two uniform random variables, the density of the sum is a triangle function. We can similarly define the convolution \( Z=W+X+Y \). It would be the convolution of the PDF of W and \( f_Z(z) \).

Propositions for Discrete RVs:

Example: \( X \sim \text{Poisson} (\lambda_1) \), \( Y \sim \text{Poisson} (\lambda_2) \)

\[
P[X+Y=n]=\sum_{k=0}^{n} \frac{e^{-\lambda_1} \lambda_1^k}{k!} \cdot \frac{e^{-\lambda_2} \lambda_2^k}{k!} = \ldots e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \left( \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} \right) = e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \left( \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} \right).
\]

That is the probability mass function of Poisson \( (\lambda_1 + \lambda_2) \)

\( X+Y \sim \text{Poisson} (\lambda_1 + \lambda_2) \)

X+Y ~ Poisson (\(\lambda_1 + \lambda_2\))
2 In-class Examples

2.1 Joint Continuous Distribution:

\[ P(X, Y) \in A = \int_A f(x, y) \, dx \, dy \]

\( f(x, y) \) is the joint probability density function

The distribution function is defined similarly: \( F(x, y) = P[X \leq x, Y \leq y] = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) \, du \, dv \)

\[ f(x, y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} \quad f_x(x) = \int_{-\infty}^{\infty} f(x, y) \, dy \quad f_y(x) = \int_{-\infty}^{\infty} f(x, y) \, dx \]

2.2 Independence of Random Variables:

Recall: Events A, B were independent if \( P[A \mid B] = P[A] \Leftrightarrow P[AB] = P[A]P[B] \)

Two random variables X, Y are independent if

\[ P[X \in A, Y \in B] = P[X \in A] P[Y \in B] = \int_A \int_B f(x, y) \, dx \, dy = \int_A \left( \int_B f(x, y) \, dy \right) \, dx \]
If $X,Y$ are independent:

- $f(x,y) = f_x(x)f_y(y)$
- $F(x,y) = F_x(x)f_y(y)$

### 2.3 Sums of Independent Random Variables

Let $X,Y$ be independent random variables $\implies f(x,y) = f_x(x)f_y(y)$. Consider $Z = X + Y$. What is $F_Z = P[Z \leq z]$?

$$P[Z \leq z] = P[X + Y \leq z] = \int \int_{x+y \leq z} f(x,y) \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f(x,y) \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_x(x)f_y(y) \, dx \, dy$$
\[ P[Z \leq z] = \int_{-\infty}^{\infty} F_x(z - y) f_y(y) \, dy = F_z(z) \]

We know that \( f_z(z) = \frac{df}{dz} F_z(z) = \frac{d}{dz} \int_{-\infty}^{\infty} F_x(z - y) f_y(y) \, dy \leftarrow \text{convolution of } f_x \text{ and } f_y. \)

If instead we wanted the density for \( V=W+X+Y \rightarrow P[V \leq u] = P[W+Z \leq u] = \int f_w(u-z)f_z(z) \, dz \rightarrow f_u = \int f_w(u-z) f_z(z) \, dz = \int_{-\infty}^{\infty} f_w(u-z) \int_{-\infty}^{\infty} f_x(z-y) f_y(y) \, dy \, dz \leftarrow \text{iterated convolution} \)

Examples:

**Gamma** \((s, \lambda) + \text{Gamma}(t, \lambda) \sim \text{Gamma}(s+t, \lambda)\)

**Normal** \(N(\mu_1, \sigma^2_1) + N(\mu_2, \sigma^2_2) \sim N(\mu_1 + \mu_2, \sigma^2_1 + \sigma^2_2)\)

**Cauchy**

\[
\begin{align*}
    f(x) &= \frac{1}{\pi} \frac{1}{1 + x^2}, \quad h(x) = \frac{1}{\pi} \frac{1}{1 + (x-\theta)^2} = \frac{\tau^2}{\pi} \frac{1}{\tau^2 + (x-\theta)^2}.
\end{align*}
\]