Student Note for 11.09 Lecture

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1 Lecture Notes

1.1 Independent Random Variables:

X,Y are independent random variables if for any two sets of real numbers A,B,

$$P\{X \epsilon A, Y \epsilon B\} = P\{X \epsilon A\} \cdot P\{Y \epsilon B\}$$

The joint CDF is equal to

$$F(a,b) = F_X(a) F_Y(b)$$

The joint probability density, f(x,y), is equal to

$$f_x(x) f_y(y)$$

If the two random variables are not independent and don't satisfy the above criteria, then we say that they are dependent.

Here is an example: $f(x,y) = 6e^{-2x} \cdot e^{-3y}, x, y \geq 0$
$f(x,y) = f_x \cdot f_y = 2e^{-2x} \cdot 3e^{-3x}$, where x and y are independent because we were able to split them up into two different functions.
Here is another function, albeit dependent. $f(x,y) = 24xy, x\epsilon(0,1), y\epsilon(0,1), x + y \leq 1$ The function is 0 otherwise. This function will lie in the region bounded between the x and y axes and x+y=1. This function can't be split into separate functions of x and y because the domain of dependence can't be split into functions of x and y. The reason for this is because the indicator function of f(x,y) is a function of x+y, not xy.

1.2 Sums of Independent Random Variables:

If X and Y are independent, let Z= X+Y. We need to calculate the PDF and CDF of Y. Its CDF is $F_Z(z) = P\{Z \leq z\} = P\{X+Y \leq z\} = \iint_D f(x,y)\,dx\,dy$ where D: $x+y \leq z$ Since X and Y are independent, we can factor. Our last equation is equal to $\iint_D f_x(x)f_y(y)\,dx\,dy$ Our domain consists of all the points to the "left" of the line x+y=z. So our equation can be rewritten as $\int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_X(x)f_Y(y)\,dx\,dy = \int_{-\infty}^{\infty} F_X(z-y)f_Y(y)\,dy$
$f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{d}{dz}\int_{-\infty}^{\infty} F_X(z-y)f_Y(y)\,dy = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)\,dy$ This is known as the convolution of f_X with f_Y This can be thought of as taking f_X and f_Y and shifting them over by z, multiplying and integrating them, and summing all of the integrals up.

Another important concept are the Independent Identically Distributed (IID) Random Variables. Consider two IID U(0,1) RVs X,Y. Z=X+Y $f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)\,dy = \int_0^1 f_X(z-y)\,dy$ Since X and Y have a range from 0 to 1, Z goes from 0 to 2. We have two cases, one in which Z is between 0 and 1, and the other where Z is between 1 and 2.
If $z \in (0,1)$, $f_X(z-y)$ is non-zero when $\int_0^z 1\,dy = z$
If $z \in (1,2)$, $f_X(z-y)$ is non-zero when $\int_{z-1}^1 1\,dy = 2-z$
So, $f_Z(z) = z$ when $z \in (0,1), 2-z$ when $z \in (1,2)$, and is 0 otherwise. When you add two uniform random variables, the density of the sum is a triangle function. We can similarly define the convolution Z= W+X+Y. It would be the convolution of the PDF of W and $f_Z(z)$

Propositions for Continuous RVs:
Proposition 1: If $X \sim \Gamma(s,\lambda)$ and $Y \sim \Gamma(t,\lambda)$, then $Z = X+Y \sim \Gamma(s+t,\lambda)$
Proposition 2: If $x_1 \sim N(\mu_1, {\sigma_1}^2)...x_n \sim N(\mu_n, {\sigma_n}^2)$ then $Y = \sum_{i=1}^n X_i \sim N(\sum_j \mu_j, \sum_j {\sigma_j}^2)$

Propositions for Discrete RVs:
Let X take values 0,1,2,... and let Y take values 0,1,2,... Then Z=X+Y takes values 0,1,2,...
P[Z=n] $= \sum_{k=0}^n P[X=k, Y=n-k] = \sum_{k=0}^n P[X=k]P[Y=n-k]$ This is a discrete convolution.

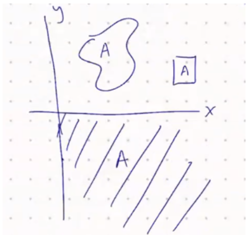
Example: $X \sim$ Poisson (λ_1), $Y \sim$ Poisson (λ_2)
P[X+Y=n]= $\sum_{k=0}^n \frac{{\lambda_1}^k e^{-\lambda_1}}{k!} \cdot \frac{{\lambda_2}^k e^{-\lambda_2}}{(n-k)!}$ =...= $e^{-(\lambda_1+\lambda_2)\sum_{k=0}^n \binom{n}{k} {\lambda_1}^k {\lambda_2}^{n-k} = e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}$
That is the probability mass function of Poisson $(\lambda_1+\lambda_2)$
X+Y $\sim$ Poisson $(\lambda_1+\lambda_2)$

2 In-class Examples

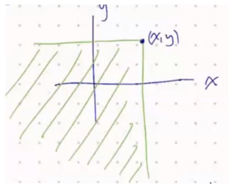
2.1 Joint Continuous Distribution:

$$P[(X,Y) \in A] = \iint_A f(x,y)\,dx\,dy$$

f(x,y) is the joint probability density function



The distribution function is defined similarly: F(x,y)=P[X≤x, Y≤y]=$\int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)\,du\,dv$



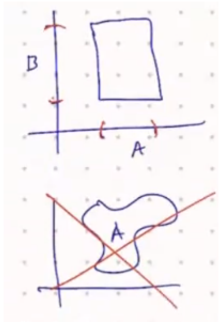
$\rightarrow f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$ $\quad f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ $\quad f_y(x) = \int_{-\infty}^{\infty} f(x,y)\,dx$

2.2 Independence of Random Variables:

Recall: Events A, B were independent if $P[A \mid B]$=P[A] ⇔ $P[AB] = P[A]P[B]$

Two random variables X,Y are independent if

$$P[X \in A, Y \in B] = P[X \in A]\,P[Y \in B] = \int_A \int_B f(x,y)\,dy\,dx = \int_A (\int_B f(x,y)\,dy)\,dx$$



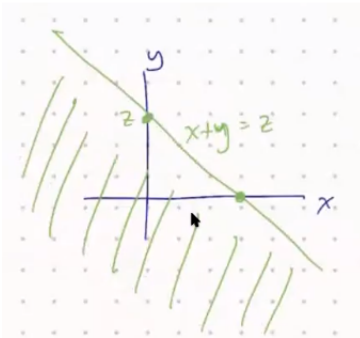
If X,Y are independent:

- f(x,y)=f$_x(x)f_y(y)$
- F(x,y)=F$_x(x)f_y(y)$

2.3 Sums of Independent Random Variables

Let X,Y be independent random variables $\Longrightarrow f(x,y) = f_x(x)f_y(y). Consider Z = X + Y. What\ is\ F_Z = P[Z \leq z]$?

$$P[Z \leq z] = P[X + Y \leq z] \Longrightarrow \iint_{x+y \leq z} f(x,y)\,dx\,dy \ = \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f(x,y)\,dx\,dy \ = \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_x(x)\,f_y(y)\,dx\,dy$$



$$P[Z \leq z] = \int_{-\infty}^{\infty} F_x(z-y) f_y(y)\, dy \;\; = F_z(z)$$

We know that $f_z(z) = \frac{d}{dz} F_z(z) = \frac{d}{dz} \int_{-\infty}^{\infty} F_x(z-y) f_y(y)\, dy = \int_{-\infty}^{\infty} f_x(z-y) f_y(y)\, dy \;\; \leftarrow convolution\, of\, f_x\, and\, f_y.$

If instead we wanted the density for V=W+X+Y $\rightarrow P[V \leq u] = P[W+Z \leq u] = \int F_w(u-z) f_z(z)\, dz \;\; \rightarrow f_u = \int f_w(u-z) f_z(z)\, dz \;\; = \int_{-\infty}^{\infty} f_w(u-z) \int_{-\infty}^{\infty} f_x(z-y) f_y(y)\, dy\, dz \;\; \longleftarrow$ *iterated convolution*

Examples:

$$Gamma(s, \lambda) \; + \; Gamma(t, \lambda) \; \sim \; Gamma(s+t, \lambda)$$

$$N(\mu_1, \sigma_1^2) + N(\mu_2, \sigma_2^2) \; \sim \; N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

$$Cauchy : f(x) = \frac{1}{\pi} * \frac{1}{1+x^2} g(x) = f(x-\theta) = \frac{1}{\pi} * \frac{1}{1+(x-\theta)^2}; \;\; h(x) = f(\frac{x-\theta}{\tau}) = \frac{1}{\pi} * \frac{1}{1+\frac{(x-\theta)^2}{\tau^2}} = \frac{\tau^2}{\pi} * \frac{1}{\tau^2 + (x-\theta)^2}$$