Math233 Note

kn1600, yz4806

November 11th 2020

1 Lecture Video

1.1 Function of Random Variable

Recall:

$$X = Normal(\mu, \sigma)$$

means that X has probability density function (PDF):

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Or:

$$P[X \leq x] = \phi(\frac{x-\mu}{\sigma})$$

Q: What if Y is a function of random variable?
A: The only way we can find PDF is through the cumulative distribution function(CDF):

$$F_Y[y] = P[Y \leq y] = P[g(x) \leq y]$$
$$=> f_Y = F_Y'$$

Example:

If X is a continous random variable with probability density f_X. Let $Y = X^2$. For $Y \geq 0$,

$$F_Y(y) = P[Y \leq y]$$
$$F_Y(y) = P[X^2 \leq y]$$
$$F_Y(y) = P[-\sqrt{y} \leq X \leq \sqrt{y}]$$
$$F_Y(y) = F(\sqrt{y}) - F(-\sqrt{y})$$
$$\frac{d}{dy}(F_Y(y) = F(\sqrt{y}) - F(-\sqrt{y}))$$
$$f_Y(y) = \frac{1}{2\sqrt{y}}[f_x(\sqrt{(y)}) + f_X(-\sqrt{y})]$$

1.1.1 General Case

Consider a continuous random variable X with PDF f_X and g a monotonic differentiable function. (i.e, g^-1 exists everywhere and $\frac{d}{dy}g^-1$ also exists)

Let Y = g(X). Then:

$$P[Y \leq y] = P[g(X) \leq y] = P[X \leq g^-1(y)]$$
$$= \int_{-\infty}^{g^-1(y)} f_X(x)dx$$

Let $z = g(x)$

$$x = g^-1(z), dx = \frac{d}{dz}(g^-1(z))dz$$
$$P[Y \leq y] = \int_{-\infty}^{y} f_X(g^-1(x))\frac{d}{dz}(g^-1(z))dz$$
$$f_Y(y) = f_X(g^-1(y))\frac{d}{dy}(g^-1(y))$$

Conclusion If X has PDF f_X and g strictly monotone and differentiable, then $Y = g(X)$ has PDF.

$$f_Y(y) = \begin{cases} f_X(g^-1(y)) \mid \frac{d}{dy}g^-1(y) \mid & if\ y = g(x) \\ 0 & otherwise \end{cases}$$

Recall $x = g^-1(y)$ is such that g(x) = y

1.2 Joint Distribution Function

The Join CDF for two random variables X,Y is function F such that:

$$F(x, y) = P[X \leq x, Y \leq y]$$

1.2.1 Discrete Case

Join Probability Mass Function:

$$p(x_i, y_j) = P[X = x_i, Y = y_j]$$
$$= P[\cup_j (X = x_i, Y = y_j)]$$
$$= \sum_j p(x_i, y_j)$$

1.3 Continous Case

$$P[X, Y \in C] = \int\int_C f(x,y)dxdy$$

$$P[X \in (a,b), Y \in (c,d)] = \int_c^d \int_a^b f(x,y)dxdy$$

$$F(x,y) = P[X \leq x, Y \leq y]$$
$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f(u,v)dudv$$

We know:

$$\frac{d^2 F}{dxdy} = f$$

Lastly

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

Since we are only integrating over R^2, this also holds for n random variables $X_1, X_2, ...X_n$

2 Class Lecture And Examples

Recall Fundamental Theorem of Calculus:

$$F(y) = \int_a^{h(y)} f(x)dx$$

$$F'(y) = f(h(y)) * h'(y)$$

Then, we can find PDF of Y = g(X) with X being a continuous random variable

$$f_Y(y) = \begin{cases} f_X(g^-1(y)) \mid \frac{d}{dy}g^-1(y) \mid & if\ y = g(x) \\ 0 & otherwise \end{cases}$$

Example 1:

Let X = Uniform(-1,1). Then:

$$f(x) = \begin{cases} 1/2 & on\ (-1,1) \\ 0 & otherwise \end{cases}$$

Let Y = X^2. We have:

$$P[Y \leq y] = P[X^2 \leq y] = P[-\sqrt{y} \leq X \leq \sqrt(y)]$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f_x(x)dx$$

$$= \int_{-\sqrt{y})}^{\sqrt{y}} \frac{1}{2}dx = \sqrt{y}$$

$$f_Y(y) = \frac{d}{dy}\sqrt{y} = \frac{1}{2\sqrt{y}}$$

Example 2:

Let X = Normal(0,1), Y = cos(X).
X $\in (-\infty, \infty)$, Y $\in$ [-1,1]

$$P[Y \leq y] = P[\cos(X) \leq y]$$
$$\neq P[X \leq \arccos(y)]$$
$$\neq \int_{-\infty}^{\arccos(y)} f(x)dx$$

This is the wrong domain. Since Y = cos(X) is an oscillating functions, we must identify the interval on which we can integrate over. We know X is a Normal Random Variable on (0,1). Hence, we can find the interval by finding the intersection of f(x) = cos(y). We can then find the right region to integrate over. So:

$$P[Y \leq y] = P[\cos(X) \leq y] = \int_{\cos(x) \leq y} f(x)dx$$

Problem 5.22 (Self Test):

Let U = Uniform(0,1), a and b are constants. a < b
a, Show that if b $\geq$ 0 then bU is uniformly distributed on (0,b) and if b < 0, then bU is uniformly distributed on (b,0)
b, Show that a + U is uniformly distributed on(a, 1 + a)
c, What function of U is uniformly distributed on (a,b)
d, Show that min(U, 1 - U) is a uniform(0,1/2) random variable
e, Show that max(U, 1 - U) is a uniform(1.2, 1) random variable
Solution:

a, Consider b > 0:

$$F(bU) = f(bU \leq x)$$
$$= f(U \leq \frac{x}{b})$$
$$= \int_0^{x/b} dx$$

$$= F(\frac{x}{b})$$

$$= \frac{x}{b}$$

$$=> f_b u(x) = F'(\frac{x}{b}) = \frac{1}{b}$$

The case b < 0 is left as exercise. b and c are also similar. d, Let Y = min(U, 1 - U)

$$F(Y) = P[\min(U, 1 - U) \leq x]$$

$$= P[U \leq x] \cup P[1 - U \leq x]$$

Since U ∩ (1-U) = ∅. We have:

$$= (\int_0^x dx) + (1 - P[U < 1 - x])$$

$$x = +(1 - (1 - x))$$

$$= 2x$$

$$=> f_u(x) = F'(y) = 2$$

Which is the density function for U = Uniform(0, 1/2). e) is similar and left as an exercise.