Random Variables and Expected Value

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Random Variables
Unlike regular variables which are set to a fixed number, *Random Variables* are not designated to a single number. In other terms, A *Random Variable* is a function defined on a sample space. An example of a random variable would be stating a value X to be equal to the result from rolling a die.

When it comes to notation, Capital letters are used to represent random variables (i.e X, Y, Z) and possible values of a random variable are lower case (i.e x, y, z)

Discrete Random Variables
A random variable that can take on at most a *countable* number of values is a discrete random variable.

The following two functions are helpful to know when dealing with random variables:
a- Probability Mass Function(*Density function*): $p(a) = P(X = a)$
b- Cumulative Distribution Function: $F(x) = P(X \leq x)$

Equation (a) states that the probability of a, p(a), is equal to the probability that X=a. Equation (b) states that the Cumulative Distribution Function of x is equal to the probability that X is less than or equal to X.
For example, referencing the example of rolling a die:
a- p(3)= P(X=3)= 1/6
b- F(3) = P(X $\leq 3) = 1/2$

Another way of writing the cumulative distribution function is in terms of the mass function which leads you to:

$$F(X) = P(X \leq x) = \sum_{x_i \leq x} p(x_i)$$

It is also worth noting that If X can take on values x_1, x_2, x_3 ... then

$$\sum_{i=1}^{\infty} p(x_i) = 1$$

Now take an example where you have a random variable X and x1,x2,x3,x4 as possible values for X. Also consider p(x1)= 1/4, p(x2)= 1/2, p(x3)=1/8, p(x4)= 1/8.
Then F(x), a piece-wise function:

$$F(x) = \begin{cases} 0 & x < x_1 \\ 1/4 & x \in [x_1, x_2) \\ 3/4 & x \in [x_2, x_3) \\ 7/8 & x \in [x_3, x_4) \\ 1 & x \in [x_4, \infty) \end{cases}$$

Expected Values:
The expected value is the weighted average of a random variable.

$$E(x) = \sum_{i=1}^{\infty} x_i P(X = x_i) = \sum_{i=1}^{\infty} x_i p(x_i)$$

Example: Rolling a die

$x_1 = 1$
$x_2 = 2$
.
.
.
$x_6 = 6$

$$P(X = x_i) = 1/6 = p(x)$$

$$E(x) = \sum_{i=1}^{6} x_i p(x_i) = 1 \times 1/6 + 2 \times 1/6 ... 6 \times 1/6 = 21/6 = 7/2$$

Example(Self-test 4.3): A coin comes up heads with probability p. Flip this coin until either heads or tails has occurred twice. Find the expected number of flips.

Let X= the number of flips until 2 heads or 2 tails

$P(X = 0 = 0$
$P(X = 1) = 0$
$P(X = 2) = P(HH) + P(TT) = p^2 + (1 - p)^2$
$P(X = 3) = P(HTT) + P(HTH) + P(THH) + P(THT) = 1 - p^2 - (1 - p)^2$
$P(X = 4) = 0$

$$\begin{aligned} E(X) &= 2 \times P(X = 2) + P(X = 3) \\ &= 2(p^2 + (1 - p)^2) + 3(1 - p^2 - (1 - p)^2) \\ &= 3 - p^2 - (1 - p)^2 \end{aligned}$$

If $p = 0, E(X) = 3 - 0 - (1 - 0)^2 = 2$
$p = 1, E(X) = 3 - 1 - 0 = 2$
$p = 1/2, E(X) = 3 - 1/4 - 1/4 = 2.5$

Additional Example 1

Q: Two cubical dice are thrown and their scores added together.

If X = "The sum of the scores on the two dice", what is P(X is divisible by 4)?

A: the Sample Space S = 2, 3, 4, 5, 6, 7, 8, 9. 10, 11, 12 and 4, 8 and 12 are the only numbers divisible by 4 in S. P(X=4)= 3/36, P(X=8)= 5/36, P(X=12)= 1/36. P(X is divisible by 4)= 1/4

Additional Example 2

Q: Two cubical dice are thrown and their scores added together.

If X= "sum of the scores on the two dice", and P(X=x)= 1/18. What is the value of x?

A: P(X=3)=1/18 and P(X=11)= 1/18. so x=3 or x=11