Conditional Probability and Bayes' Rule
3.1-3.3

September 23 2020

1 Lecture Video

Materials and problems covered in the lecture video. To be completed by Robert.

Introduction

P(die comes up = 3) = $\frac{1}{6}$
Now imagine that you know that the die comes up odd:

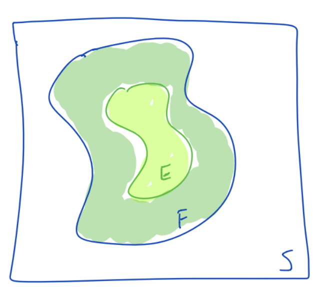
$$\text{P(die} = 3 \mid \text{die comes up odd)} = \frac{1}{3}$$

(| ⟶ notation for "given". Anything to the right of the bar is used to compute probability by restricting the set of allowable events)

Conditional Probability

Definition: The conditional probability of E **given** F (assuming P(F) > 0) is:

$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$



Example (Figure 1):

P(F) = .5 *What is the probability of $P(E \cap F)$?*
P(E) = .25 P(E ∩ F) = ? = P(E) = .25 since E ⊂ F

P(E | F) = $\frac{.25}{.5}$ = .5
P(E | F) = $\frac{P(E \cap F)}{P(F)}$ ⟶ likewise, **P(F | E)** = $\frac{P(E \cap F)}{P(F)}$

Bayes' Rule

After rearranging, we can see that:

$$P(E \cap F) = P(E \mid F)P(F) \quad \text{They are equal!}$$
$$= P(F \mid E)P(E)$$

Setting them equal to each other results in:

P(E | F) P(F) = P(F | E) P(E)

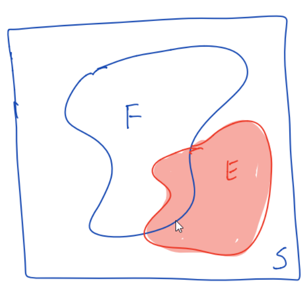
⟶ P(E | F) = $\frac{P(F|E)P(E)}{P(F)}$ where P(F | E) P(E) equals to P(E ∩ F)

This is Bayes' Rule.
More generally...

Multiplication Rule

P(E F) = P(F | E) P(E) ⟶ = P(E | F) P(F)

More general case:
$P(E_1E_2E_3...E_n) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1E_2)...P(E_n \mid E_1 \cdots E_{n-1})$
One example of the multiplication rule's applicability is Bayes' Rule, as it is the numerator in that equation.



Law of Probability

The sample space (such as the one in Figure 2. above) can be split into two **mutually exclusive** events: $S \cup E^c$, $E \cap E^c \neq \emptyset$

$F = (E \cap F) \cup (E^c \cap F)$

> P(F) = P(EF) + $P(E^cF)$
> where $(E \cap F)$ and $(E^c \cap F)$ are mutually exclusive

More generally, if $E_1, ..., E_n$ are mutually exclusive events and $E_1 \cup E_2 \cup ... \cup E_n = S$, then for any event F,

$$\bigcup_{i=1}^{n}(F \cap E_i) \longrightarrow P(F) = \sum_{i=1}^{n} P(FE_i)$$

Often very useful in Bayes' Rule:

$$\text{P(E | F)} = \frac{P(F|E)P(E)}{P(F)} = \frac{P(F|E)P(E)}{P(EF)+P(E^cF}$$

Expand these terms using conditional probability

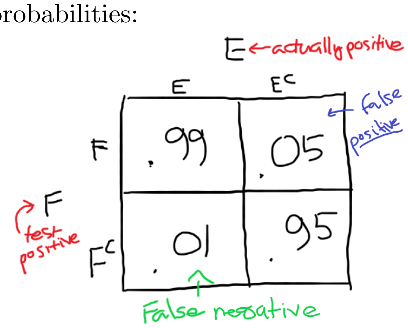
$$= \frac{P(F|E)P(E)}{P(E|F)P(F)+P(E^c|F)P(F)}$$

$$\text{Since P(EF)} = P(E \mid F)P(F) = P(F \mid E)P(E)$$
$$\longrightarrow \frac{P(F|E)P(E)}{P(F|E)P(E)+P(F|E^c)P(E^c)}$$

Example
Let E = event you are actually Covid-positive
F = event you test positive.

$$P(E \mid F) = \frac{P(F|E)P(E)}{P(F)} = \frac{P(F|E)P(E)}{P(FE)+P(FE^c)} = \frac{P(F|E)P(E)}{P(F|E)P(E)+P(F|E^c)P(E^c)}$$

Let's consider their probabilities:



$$= \frac{.99 \times .0001}{.99 \times .0001 + .05 \times .9999} = \frac{.000099}{.000099+.049995} = \frac{.000099}{.050094} \approx \boxed{.002}$$

The final answer here signifies the actual positive rate of Covid! (based on the ex. data)

Not exactly: The final answer is the conditional probability that you are COVID position GIVEN that you test positive. It is so low because of the large false positive rate relative to the background rate of incidence, 0.0001.

2 In Class Examples

Problem 3.51 Prostate cancer is the most common type of cancer found in males. As an indicator of whether a male has prostate cancer, doctors often perform a test that measures the level of the prostate-specific antigen (PSA) that is produced only by the prostate gland. Although PSA levels are indicative of cancer, the test is notoriously unreliable. Indeed, the probability that a noncancerous man will have an elevated PSA level is approximately .135, increasing to approximately .268 if the man does have cancer. If, on the basis of other factors, a physician is 70 percent certain that a male has prostate cancer, what is the conditional probability that he has the cancer given that

(a) the test indicated an elevated PSA level?
(b) the test did not indicate an elevated PSA level?

Solution Let C be the event that male has cancer and T be the event that PSA level is elevated.
(a) The conditional probability that he has the cancer given that the test indicated an elevated PSA level is

$$\begin{aligned} P(C \mid T) &= \frac{P(T \mid C)P(C)}{P(T)} \\ &= \frac{P(T \mid C)P(C)}{P(TC) + P(TC^c)} \\ &= \frac{P(T \mid C)P(C)}{P(T \mid C)P(C) + P(T \mid C^c)P(C^c)} \\ &= \frac{(0.268)(0.7)}{(0.268)(0.7) + (0.125)(1 - 0.7)} \\ &= \frac{0.1876}{0.2281} \\ &= 0.8224 \end{aligned}$$

(b) The conditional probability that he has the cancer given that the test did not indicate an elevated PSA level is

$$\begin{aligned} P(C \mid T^c) &= \frac{P(T^c \mid C)P(C)}{P(T^c)} \\ &= \frac{P(T^c \mid C)P(C)}{1 - P(T)} \\ &= \frac{P(T^c \mid C)P(C)}{1 - P(T)} \\ &= \frac{(1 - 0.268)(0.7)}{1 - 0.2281} \end{aligned}$$

$$= \frac{0.5124}{0.7719}$$
$$= 0.6638$$

Theoretical Exercise 3.2 Let $A \subset B$. Express the following probabilities as simply as possible:

$$P(A \mid B), P(A \mid B^c), P(B \mid A), P(B \mid A^c)$$

Solution Since $A \subset B$, the Venn diagram of their relationship is

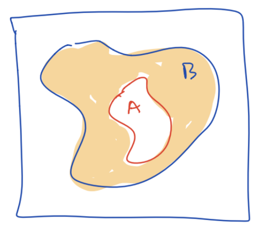


Figure 1: $A \subset B$

1. $P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)}$
2. $P(A \mid B^c) = \frac{P(AB^c)}{P(B^c)} = \frac{\emptyset}{P(B^c)} = 0$
3. $P(B \mid A) = \frac{P(AB)}{P(A)} = \frac{P(A)}{P(A)} = 1$
4. $P(B \mid A^c) = \frac{P(A^cB)}{P(A^c)} = \frac{P(B)-P(AB)}{P(A^c)} = \frac{P(B)-P(A)}{P(A^c)}$

Theoretical Exercise 3.5
(a) Prove that if E and F are mutually exclusive, then

$$P(E \mid E \cup F) = \frac{P(E)}{P(E) + P(F)}$$

(b) Prove that if $E_i, i \geq 1$ are mutually exclusive, then

$$P(E_j \mid \bigcup_{i=1}^{\infty} E_i) = \frac{P(E_j)}{\sum_{i=1}^{\infty} P(E_i)}$$

Solution (a) According to the definition of conditional probability,

$$P(E \mid E \cup F) = \frac{P(E \cap (E \cup F))}{P(E \cup F)}$$

Because E and F are mutually exclusive, according to the probability axiom of mutually exclusive sets,

$$\frac{P(E \cap (E \cup F))}{P(E \cup F)} = \frac{P(E)}{P(E) + P(F)}$$

(b) According to the definition of conditional probability,

$$P(E_j \mid \bigcup_{i=1}^{\infty} E_i) = \frac{P(E_j \cap (\bigcup_{i=1}^{\infty} E_i))}{P(\bigcup_{i=1}^{\infty} E_i)}$$

Because $E_i, i \geq 1$ are mutually exclusive, according to the probability axiom of mutually exclusive sets,

$$\frac{P(E_j \cap (\bigcup_{i=1}^{\infty} E_i))}{P(\bigcup_{i=1}^{\infty} E_i)} = \frac{P(E_j)}{\sum_{i=1}^{\infty} P(E_i)}$$

3 Additional Problems

Additional problems for this class. To be completed by Zheyue.

Example 1 Let a pair of fair dice be tossed. If the sum is 6, find the probability that one of the dice is a 2.

Solution The equation for conditional probability is

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

In this case $P(F)$ is considered to be the probability that the two rolls yields a sum of 6, and $P(E \cap F)$ is the probability of yielding a sum of 6 and have one of the die is a 2. Now E = Sum is 6 = $(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)$.Out of those, only two of them contains a 2. $(2, 4), (4, 2)$ thus$P(E \mid F) = \frac{2}{5}$

Example 2 Four people, called North, South, East and West, are each dealt 13 cards from an ordinary deck of 52 cards.
(a) If South has no aces, find the probability p that his partner North has exactly two aces.

Solution(a) since South has no Aces, then we know that there are 39 cards, including 4 aces, divided among North, East and West. Thus the total amount of ways that North can be dealt 13 cards are $\binom{39}{13}$

$$P(E) = \binom{39}{13}$$

Now consider the probability of

$$P(E \cap F)$$

There are four aces in the deck, so there are a total of $\binom{4}{2}$ ways of getting 2 Aces, and a total of $\binom{35}{11}$ ways of getting 11 cards from the remaining 35 cards Thus the solution to part(a) is

$$p = \frac{P(E \cap F)}{P(F)} = \frac{\binom{4}{2}\binom{35}{11}}{\binom{39}{13}} = \frac{650}{2109}$$

Example 3 In a certain college, 25 percent of the students failed mathematics, 15 percent of the students failed chemistry, and 10 percent of the students failed both mathematics and chemistry. A student is selected at random.

(a) If he failed chemistry, what is the probability that he failed mathematics?
(b) If he failed mathematics, what is the probability that he failed chemistry?

Solution(a) Let the probability of a student failing Math be M, and failing Chemistry be C hence

$$P(M) = 0.25$$
$$P(C) = 0.15$$
$$P(M \cap C) = 0.1$$

hence the probability of a student failing Math, given that he failed Chem is

$$p(M \mid C) = \frac{P(M \cap C)}{P(C)} = \frac{0.1}{0.15} = \frac{2}{3}$$

Solution(b) Continuing with the same idea in part(a)

$$p(C \mid M) = \frac{P(C \cap M)}{P(M)} = \frac{0.1}{0.52} = \frac{2}{5}$$

Example 4 Three machines A, B and C produce respectively 60 percent, 30 percent and 10 percent of the total number of items of a factory. The percentages of defective output of these machines are respectively 2 percent, 3 percent and 4 percent. An item is selected at random and is found defective. Find the probability that the item was produced by machine C.

Let X = defective items. We seek $P(C \mid X)$ It is difficult to compute this directly because it is difficult to figure out $P(X)$, aka the overall percentage of productions that are defects. Instead we can apply Bayes' theorem which gives us

$$\begin{aligned} p(C \mid X) &= \frac{P(C)P(X \mid C)}{P(A)P(X \mid A) + P(B)P(X \mid B) + P(C)P(X \mid C)} \\ &= \frac{(0.1)(0.04)}{(0.6)(0.02) + (0.3)(0.03) + (0.1)(0.04)} \\ &= \frac{4}{25} \end{aligned}$$