1 Multinomial coefficients

Multinomial coefficient is an analogous of binomial coefficient in that they correspond directly to the counting problems.

Imagine we have N distinct objects:

\[ 0_1, 0_2, ..., 0_N \]

sort into M bins: \( N_i \) objects into bin \( i \) (Order of each bin does not matter)

\[ \sum_{i=1}^{M} n_i = N \]

\[
\binom{N}{n_1} \times \binom{N-n_1}{n_2} \times \binom{N-n_1-n_2}{n_3} \times ... \times \binom{N-n_1-...-n_{M-1}}{n_M}
\]

(Filling bin 1) \* (Filling bin 2) \* (Filling bin 3) \* ... \* (Filling bin M)

\[
= \frac{N!}{n_1!(N-n_1)!} \times \frac{(N-n_1)!}{n_2!(N-n_1-n_2)!} \times \frac{(N-n_1-n_2)!}{n_3!(N-n_1-n_2-n_3)!} \times ... \times \frac{n_M!}{n_M!} \times 1!
\]

\[
= \frac{N!}{n_1!n_2!n_3!...n_M!} = \binom{N}{n_1, n_2, n_3, ..., n_M} (\text{Multinomial Coefficient})
\]

Multinomial Coefficient is the number of ways to sort N distinct objects into bins, each having \( n_i \) objects.

2 Extension to Multinomial Theorem

\[
(x_1 + x_2 + ... + x_r)^n = \sum_{(n_1, n_2, ..., n_r)} x_1^{n_1} x_2^{n_2} \times ... \times x_r^{n_r}
\]

Here, \( \Sigma \) is sum over \( n_1, n_2, ..., n_r \) such that \( n_1 + n_2 + ... + n_r = n \)
3 Sample Spaces and Events

Sample space (S) is the set of possible outcomes of an experiment.

Ex: Roll two dice

\[ S = \{(1,1), (2,1), (3,1), \ldots, (6,6)\} \]

A collection of outcomes is called an event. An event is a subset of S.

Ex: Let E and F be two events defined on the sample space S.
1-1) Define a new event \( G = E \cup F \) where E is a set of the first die = 2, and F is a set of the second die less than 3.

Here, \( \cup \) is all outcomes that are contained in E or F.

\[ E = \{(2,1), (2,2), \ldots, (2,6)\} \]
\[ F = \{(1,1), (2,1), (3,1), \ldots(6,1), (1,2), (2,2), \ldots, (6,2)\} \]

Therefore, \( G = E \cup F = \{(2,1), (2,2), \ldots, (2,6), (1,1), (1,2), (3,1), (3,2), \ldots, (6,1), (6,2)\} \)

1-2) Define another new event \( H = E \cap F \)

Here, \( \cap \) is outcomes that are contained in both E and F.

Therefore, \( H = E \cap F = \{(2,1), (2,2)\} \)

**Definition:** Mutually Exclusive

We say E and F are mutually exclusive if

\[ E \cap F = \{\} \] (empty set)
\[ = \emptyset \] (the null set)

Other properties of set theory useful for probability

Multiple events: \( G = \bigcup_{i=1}^{n} E_i \)

Complement: \( E^c = \) all outcomes that are not in E

\[ U = E \cup E^c, E \cap E^c = \emptyset \]

Here, Yellow part is: \( A \cup B \), Blue part is: \( A \cap C \)
4 Set Operation Laws

- **Commutative Law**: \( E \cup F = F \cup E, \ E \cap F = F \cap E \)

- **Associative Law**: \((E \cup F) \cup G = E \cup (F \cup G), (E \cap F) \cap G = E \cap (F \cap G)\)

- **Distributive Law**: \((E \cup F) \cap G = (E \cap G) \cup (F \cap G)\)

4.1 DeMorgan’s Laws

**DeMorgan’s Law**: the following laws could be combined into DeMorgan’s Laws as such:

\[
\left( \bigcup_{i=1}^{n} E_i \right)^c = \bigcap_{i=1}^{n} E_i^c \\
\left( \bigcap_{i=1}^{n} E_i \right)^c = \bigcup_{i=1}^{n} E_i^c
\]

Hint:

- Multinomial Coefficient: number of ways to group \( n \) objects into groups of \( n_1, n_2, \ldots, n_r \) object, with \( n_1 + n_2 + \ldots + n_r = n \).

- Binomial Coefficient: number of ways to group \( n \) objects into groups of \( n_1, n_2 \) object, with \( n_1 = k \) and \( n_2 = n - k \).

5 Graphical representation of DeMorgan’s laws

1. \( \left( \bigcup_{i=1}^{n} E_i \right)^c = \bigcap_{i=1}^{n} E_i^c \)
2. \((\cap_{i=1}^{n} E_i)^c = \cup_{i=1}^{n} E_i^c\)

6 Example problems

Example 1

Prove that if \(E \subset F\), then \(F^c \subset E^c\).

This can be proved with a Venn diagram.

We can see that \(F^c \subset E^c\)
Example 2

Simplify \((E \cup F)(E \cup F^c)\) using a Venn Diagram

We can see that the common area is exactly \(E\)

7 Textbook Example:

1. Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

Solution: There are \(\frac{10!}{5!5!} = 252\) possible divisions.

2. In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

Solution: Note that this example is different from the previous example because now the order of the two teams is irrelevant. That is, there is no A and B team, but just a division consisting of 2 groups of 5 each. Hence, the desired answer is \(\frac{10! / (55)}{2!} = 126\).