Lecture 1

September 9th 2020

1 Counting

① Roll a die: Outcomes are 1,2,3,4,5,6
② Roll a die, choose a and card from playing deck:
Die: 6 possibilities
Cards: 52 distinct cards.
Total number of die-card pairs: $6 \cdot 52 = 312 possibilities$

In general, if we have M experiments, each with n_i distinct outcomes, then we have $n_1 \cdot n_2 \cdot n_3 \cdot n_4 \cdot ... \cdot n_M$ total possible outcomes.
Back to our example,
$n_1 = 6$
$n_2 = 52$

2 Permutations – ordering distinct object.

EXAMPLE:
How many ways can I order 6 numbers: Fill in the blanks :
$\underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 6! = 720 \leftarrow possibilities$

$Factorial$:n! = n $\cdot \cdot (n-2) \cdot ... \cdot (n-(n-1)) = \prod_{j=1}^{n} j$

Convention: $0! = 1$

EXAMPLE:
11 people on a soccer team, each person plays one position.
$11 \cdot 10 \cdot ... \cdot 2 \cdot 1 = 11! = 39,916,800$

EXAMPLE:
Given 4 textbooks on math, 3 on English.
7 total textbooks means 7! possible orderings of all 7 textbooks

Sub-example If all math textbooks come first, then English textbooks:
$\underline{M} \cdot \underline{M} \cdot \underline{M} \cdot \underline{M} \cdot \underline{E} \cdot \underline{E} \cdot \underline{E} \Rightarrow 4! * 3! = 24 \cdot 6 = 144$ orderings

EXAMPLE:

How many orderings of the letters in PEPPER are there? If each letter is assumed to be distinct, them are $6! = 720$ possibilities.

If the E's are <u>not</u> distinct, then we must divide 720 by the number of permutation of the E's:

$$\frac{720}{2!} = 360 = \frac{6!}{2!}$$

Likewise, if the P's are not distinct, then we must divide by the number of permutations of this letter : 3!

So the number of <u>distinct</u> words that can be made using the letters in PEPPER is:

$$\frac{6!}{2!\cdot 3!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1} = 60$$

In general, if we have n objects, of which n_1 are alike, n_2 are alike, $\ldots n_r$ are alike, then the number of <u>distinct</u> permutations is

$$\frac{n!}{n_1!\cdot n_2!\cdot\ldots\cdot n_r!}$$

3 Combinations – How many distinct unordered groups can be found from a set of objects.

EXAMPLE:

Take the letters A,B,C,D,E.

How many groups of 3 letter can be form?

$$\frac{5\cdot4\cdot3}{3!} = \frac{60}{6} = 10$$

In general, the number of unordered sets of r distinct object that can be formed from n distinct elements is

$$\binom{n}{r} = \frac{n\cdot(n-1)\cdot(n-2)\ldots(n-r+1)}{r!}\cdot\frac{(n-1)\cdot(n-r-1)\ldots1}{(n-r)\cdot\ldots\cdot1} = \frac{n!}{(n-r)!}\cdot\frac{1}{r!} = \frac{n!}{(n-r)!\cdot r!}$$

we usually call it ”n choose r’

<u>APPLICATION: Binomial Theorem</u>

<u>Binomial Theorem</u>: ~~$(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k}$~~ (*) $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$

Ex: $(x+y)^2 = \binom{2}{0}x^2 + \binom{2}{1}2xy + \binom{2}{2}y^2$

<u>Proof</u> (By induction):

① prove for $n = 0, 1, 2$ we prove it by calculation

② Assume (*) is true for $n = m$

③ Show that (*) also holds for $n = m + 1$

We show it as follow:

$$(x+y)^{m+1} = (x+y)(x+y)^m = \sum_{k=0}^{m+1} \binom{m+1}{k} x^k \cdot y^{m+1-k}$$

$$= \text{(by assumption)}$$

$$= (x+y) \sum_{k=0}^{m} \binom{m}{k} x^k \cdot y^{m-k}$$

$$= \sum_{k=0}^{m} \binom{m}{k} x^{k+1} \cdot y^{m-k} + \sum_{k=0}^{m} \binom{m}{k} x^k \cdot y^{m+1-k}$$

$$= \sum_{k=1}^{m+1} \binom{m}{k-1} x^k \cdot y^{m-(k-1)} + \sum_{k=0}^{m} \binom{m}{k} x^k \cdot y^{m+1-k}$$

$$= x^{m+1} + \sum_{k=1}^{m} \binom{m}{k-1} x^k \cdot y^{m+1-k} + y^{m+1} + \sum_{k=1}^{m} \binom{m}{k} x^k \cdot y^{m+1-k}$$

$$= x^{m+1} + \sum_{k=1}^{m} (\binom{m}{k-1} + \binom{m}{k}) x^k \cdot y^{m+1-k} + y^{m+1}$$

And

$$\begin{aligned}\tbinom{m}{k-1} + \tbinom{m}{k} &= \tfrac{m!}{(m-(n-1))!} \cdot \tfrac{m!}{(m-k)!k!} \\ &= \tfrac{m! \cdot k}{(m+1-k)!k!} + \tfrac{m!(m+1-k)}{(m+1-k)!k!} \\ &= \tfrac{m! \cdot k + m!(m+1) - m! \cdot k}{(m+1-k)!k!} \\ &= \tfrac{(m+1)!}{(m+1-k)!k!} \\ &= \tbinom{m+1}{k}\end{aligned}$$

In the end we have:

$$x^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} x^k \cdot y^{m+1-k} + y^{m+1}$$

$$= \tbinom{m+1}{m+1} x^{m+1} \cdot y^0 + \sum_{k=1}^{m} \binom{m+1}{k} x^k \cdot y^{m+1-k} + \binom{m+1}{0} x^0 \cdot y^{m+1-0}$$

$$= \sum_{k=0}^{m+1} \binom{m+1}{k} x^k \cdot y^{m+1-k}$$

Till here proving is finished.

Zoom Lecture Example:

Part I:

A bug wants to go from A to B (shown in Figure 1), how many different paths for it to get there?

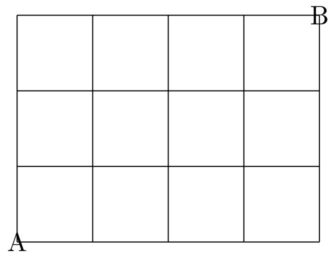

Figure.1

From A to B, we need to move 4 rights (R) and 3 up (U), and there are seven moves in total. For example, one way to reach B is URURURR. There are two ways to solve this:

- **Combination**
  The problem can be transformed into asking how many different ways for 4Rs and 3Us to fit in 7 slots, so we first want to choose 4 slots for R, then the rest is for Us by default.
  We have

$$\binom{7}{4} = \frac{7!}{3! \times 4!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$$

- **Permutation**
  The problem can be transformed into how many different ways 4Rs and 3Us can be ordered (note that R is repeated four times, and U is repeated 3 times). Then we have

$$\frac{7!}{3! \times 4!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$$

Part II:
We still have the same graph, but this time there is a point C in the middle of A and B. The question is how many different paths from A to B that goes through C?

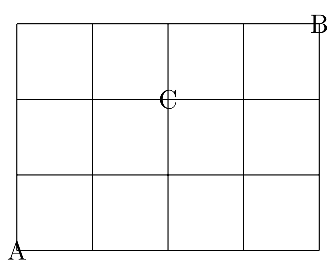

Figure.2

In this problem, we need to find the number of paths(M) from A to C, then we need find the number of paths(N) from C to B, then we multiply M by N.

For M, there are two Rs and two Us in total, therefore by using the same method from above, we get:

$$M = \binom{4}{2} = \frac{4!}{2! \times 2!} = \frac{4 \times 3}{2 \times 1} = 6$$

For N, there are two Rs and one U, therefore by using the same method, we get:

$$N = \binom{3}{2} = \frac{3!}{2! \times 1!} = \frac{3}{1} = 3$$

$$M \times N = 6 \times 3 = 18$$

[Note: $\binom{n}{k} = \binom{n}{n-k}$, and $\binom{3}{1} = \binom{3}{2}$]