Conditional Expectation

$$E[X \mid Y=y] = \int x \, f_{X|Y}(x,y)\, dx$$ ⟹ going to depend on y.

$$\underbrace{E\big[E[X \mid Y]\big]}_{\text{this entire quantity is a random variable.}} = \int E[X \mid Y=y]\, f_Y(y)\, dy$$

$$= \int \int x \, \underbrace{f_{X|Y}(x,y)}_{\frac{f(x,y)}{f_Y(y)}}\, dx \; f_Y(y)\, dy$$

$$= \iint x \, \cancel{f_Y(y)} \, \frac{f(x,y)}{\cancel{f_Y(y)}} \, dx\, dy$$

$$= \iint x \, f(x,y)\, dx\, dy$$

$$= E[X].$$

$$\boxed{E\big[E[X \mid Y]\big] = E[X]}$$

Example

independent

N = number of customers entering a store.

X_i = amount of money customer i spends (IID)

$$T = \sum_{i=1}^{N} X_i$$

What is the expected value of T?

$$E[T] = E\left[\sum_{i=1}^{N} X_i\right]$$

$$= E[\,E[T \mid N]\,]$$

$$= E\left[\,E\left[\sum_{1}^{N} X_i \mid N\right]\right]$$

use this idea here

We can write:

$$E\left[\sum_{1}^{N} X_i \mid N = n\right] = E\left[\sum_{1}^{n} X_i\right]$$

$$= n\,E[X_i]$$

$$= E\left[\underbrace{N}_{\text{R.V}}\ \underbrace{E[X_i]}_{\text{a number}}\right]$$

$$= E[N] \cdot E[X_i]$$

Conditional Variance

$$Var[X|Y] = E[(X - E[X|Y])^2 | Y]$$

$$\Rightarrow Var(X) = E[Var[X|Y]] + Var[E[X|Y]].$$

Example

Let $N > 0$ be an integer-valued r.v.

X_i are IID, and N, X_i are independent

$$T = \sum_{i=1}^{N} X_i$$

$$Var[T] = Var[\sum_i^N X_i]$$

$$= \underbrace{E[Var[\underbrace{\sum_i^N X_i}_{T} | N]]} + \underbrace{Var[E[\sum_i^N X_i | N]]}_{Var[N\, E[X_i]]}$$

$$= E[N \underbrace{Var(X_i)}_{\text{a number}}] + Var[N \underbrace{E[X_i]}_{\text{a number}}]$$

$$= Var[X_i]\, E[N] + (E[X_i])^2\, Var[N].$$