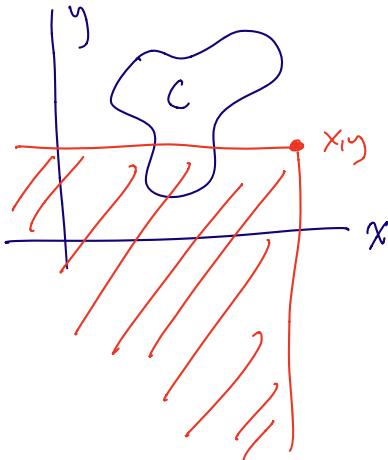


Theory of Probability

Nov 4, 2020

$$P[(X, Y) \in C] = \iint_C f(x, y) dx dy.$$



$$F(x, y) = P[X \leq x, Y \leq y]$$

$$= \iint_{-\infty}^y \int_{-\infty}^x f(u, v) du dv$$

$$f(x, y) \sim \text{pdf of } X, Y$$

$$f_x \sim \text{pdf of } X$$

$$f_x(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

$$\frac{\partial^2 F}{\partial x \partial y} = f$$

Functions of a continuous random variable:

If f is the pdf of X , F the CDF, and $Y = g(x)$, then what is the pdf/CDF of Y ?

$$\begin{aligned} P[Y \leq y] &= P[g(x) \leq y] = P[\underline{x} \leq \underline{g}(y)] \\ &= \int_{-\infty}^{g(y)} f(x) dx = F_g(y) \end{aligned}$$

CDF of Y

1

We know that

$$f_y(y) = \frac{d}{dy} F_Y(y)$$

Fundamental theorem of calculus:

$$F(y) = \int_a^{h(y)} f(x) dx$$

What is F' ? $\Rightarrow f(h(y)) h'(y)$

$$F_y(y) = \int_{-\infty}^{g^{-1}(y)} f(x) dx$$

$$= \int_{-\infty}^y f(g^{-1}(z)) \frac{dg^{-1}}{dz}(z) dz$$

i) the pdf of Y .

$$X \sim \text{Uniform}(-1, 1) \Rightarrow f_x(x) = \begin{cases} \frac{1}{2} & \text{on } (-1, 1) \\ 0 & \text{otherwise} \end{cases}$$

$$Y = X^2$$
A hand-drawn graph of the parabola $y = x^2$. The curve opens upwards with its vertex at the origin (0,0). It passes through points such as (-1, 1), (1, 1), and (2, 4). The x-axis is labeled with -1 and 1, and the y-axis is implied by the curve's symmetry.

By previous calculation :

$$P[Y \leq y] = P[X^2 \leq y] = \cancel{P[X \leq \sqrt{y}]}$$

$$= P[-y \leq X \leq y] \quad y \geq 0$$

$$= \int_{-y}^y f(x) dx .$$

Monotonic Increasing.

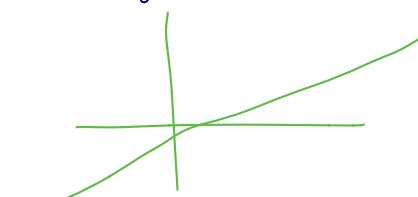
If $y > x$, then
 $g(y) > g(x)$.

Monotonic non decreasing:
 If $y > x$, then
 $g(y) \geq g(x)$.

$$g(x) = z$$

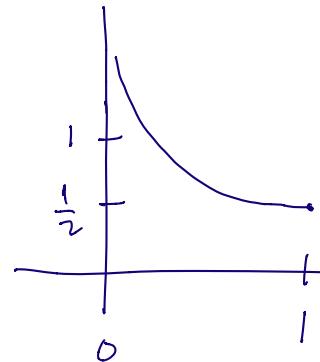
$$\text{Let } x = j(z) \quad x: -\infty \rightarrow j(y) \\ z: -\infty \rightarrow y$$

(Assume g is monotonic increasing and differentiable)



$$X \sim \text{Uniform}(-1, 1) \Rightarrow f_X(x) = \begin{cases} \frac{1}{2} & \text{on } (-1, 1) \\ 0 & \text{otherwise} \end{cases}$$

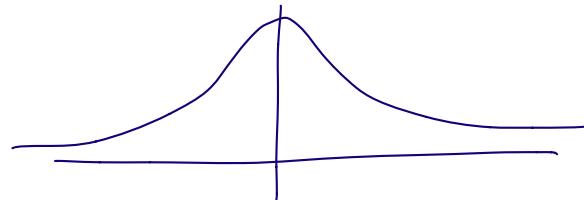
$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} dx = \left[\frac{x}{2} \right]_{-\sqrt{y}}^{\sqrt{y}} = \sqrt{y}$$



$$f_y(y) = \frac{d}{dy} (\sqrt{y}) = \frac{1}{2\sqrt{y}}$$

$$\begin{aligned} \int_0^1 f_y(y) dy &= \int_0^1 \frac{1}{2} \frac{1}{\sqrt{y}} dy \\ &= \frac{1}{2} \left[2\sqrt{y} \right]_0^1 = 1 \end{aligned}$$

Let $X \sim N(0, 1)$



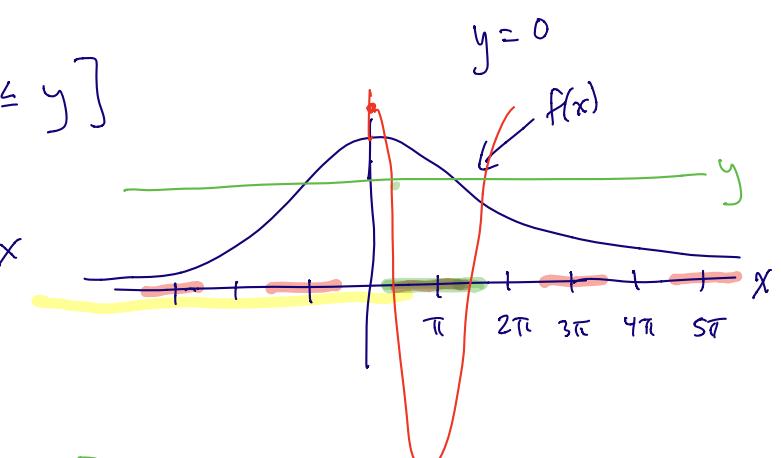
$$\begin{aligned} Y &= \cos X \\ &= g(x) \end{aligned}$$

$$X \in (-\infty, \infty)$$

$$Y \in [-1, 1]$$

$$P[Y \leq y] = P[\cos X \leq y]$$

$$= \int_{\cos x \leq y} f(x) dx$$



$$\neq P[X \leq \arccos y]$$

$$\neq \int_{-\infty}^{\arccos y} f(x) dx$$