Theory of Probability $\quad X \sim \text{Poisson}(\lambda)$. $\quad$ Oct 19, 2020

$$P[X=k] = e^{-\lambda}\frac{\lambda^k}{k!}$$ when $\lambda > 0$ is some parameter.

$E[X] = \lambda$

$Var[X] = \lambda$.

If $Y \sim \text{Binomial}(n,p)$ random variable, and if $n$ is large and $p$ is small, with $np = \lambda \sim \mathcal{O}(1)$, then

$Y$ is approximately $\text{Poisson}(\lambda)$.

$P[Y=k] \approx P[X=k]$ when $X \sim \text{Poisson}(\lambda)$.

$$\boxed{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}} = e^{\lambda}.$$

Theoretical Exercise 4.17

$X \sim \text{Poisson}(\lambda)$.

Show $P[X=k]$ increases monotonically, then decreases monotonically as a function of $k$.

① Show that $\dfrac{P[X=k+1]}{P[X=k]} > 1$

$$\frac{P[X=k+1]}{P[X=k]} = \frac{\cancel{e^{-\lambda}} \dfrac{\lambda^{(k+1)}}{(k+1)!}}{\cancel{e^{-\lambda}} \cancel{\dfrac{\lambda^k}{k!}}} = \frac{\lambda}{k+1}$$

$$\frac{\lambda}{k+1} > 1 \quad \text{if} \quad k+1 < \lambda$$

$$= k < \lambda - 1$$

<u>Self-test Exercise 4.14</u>

On average, 5.2 hurricanes hit a certain region every year.

What is the probability that there will be 3 or fewer hurricanes this year?

Let $X$ = # of hurricanes this year, then

$X \sim \text{Poisson}(\lambda)$, and $\lambda = 5.2$.

$X \sim \text{Poisson}(\lambda t)$, with $t = 1$ year, $\lambda = 5.2/\text{year}$.

$$P[X \le 3] = P[X=0] + P[X=1] + P[X=2] + P[X=3]$$

$$= \sum_{k=0}^{3} e^{-\lambda} \frac{\lambda^k}{k!} = \approx 24\%$$

## Self-test exercise 4.15

$X \sim \text{Poisson}(\lambda)$ which models the number of eggs laid on a leaf by a certain insect.

Let $Y =$ positive values of $X$

$$P[Y=k] = P[X=k \mid X>0]$$

What is $E[Y]$.

$$E[Y] = \sum_{k=1}^{\infty} k \, P[Y=k]$$

$$= \sum_{k=1}^{\infty} k \, P[X=k \mid X>0]$$

$$= \sum_{k=1}^{\infty} k \, \frac{P[X=k \cap X>0]}{P[X>0]}$$

$P[X=k \cap X>0]$
$= P[X=k]$ if $k>0$

$P[X>0] = 1 - P[X=0]$
$= 1 - e^{-\lambda} \frac{\lambda^0}{0!}$
$= 1 - e^{-\lambda}$

$$= \frac{1}{1-e^{-\lambda}} \sum_{k=1}^{\infty} k \, P[X=k]$$

$$= \frac{1}{1-e^{-\lambda}} \underbrace{\sum_{k=0}^{\infty} k \, P[X=k]}_{E[X]}$$

$$= \frac{E[X]}{1-e^{-\lambda}}$$

$$= \frac{\lambda}{1-e^{-\lambda}}$$