

# Theory of Probability

Oct 5, 2020

## Random Variables

Discrete random variables :

$X$  is a discrete random var. if it can take on a countable set of values

$$X_1, X_2, \dots \rightarrow$$

Probability mass function  $p(x_i) = P(X = x_i)$   
(density)

(Cumulative) Distribution function

$$F(x) = P(X \leq x)$$

$$= \sum_{x_i \leq x} p(x_i)$$

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-  $F$  is non-decreasing

$$\Rightarrow F(y) \geq F(x) \quad \text{if } y > x$$

Example: Dic  $P(X = 1) = \frac{1}{6}$

$$P(X = 6) = \frac{1}{6}$$

$$F(3) = P(X \leq 3) = \frac{1}{2}$$

$$F(3.5) = P(X \leq 3.5) = \frac{3}{2}$$

Expected value:

$$E(X) = \sum_{i=1}^{\infty} x_i \cdot P(X = x_i)$$

$$= \sum_{i=1}^{\infty} x_i p(x_i)$$

Theoretical Ex: 4.2

If  $X$  has distribution function  $F$ ,  $P(X \leq x) = F(x)$ , then what is the distribution function of  $e^X$ ?

$\Rightarrow Y = e^X$  is another random variable

$$G(y) = P(Y \leq y)$$

$$= P(e^X \leq y)$$

$$= P(X \leq \log y) = F(\log y)$$

$$\underbrace{F(\log y)}$$

$\Rightarrow e^X = Y$  has distribution function  $G(y) = F(\log y)$ .

Self-test Ex: 4.3

A coin comes up heads with probability  $p$ .

Flip this coin until either heads or tails has occurred twice. Find the expected number of flips.

Let  $X$  = the number of flips until 2 heads or 2 tails.

$$P(X=0) = 0$$

$$P(X=1) = 0$$

$$P(X=2) = P(HH) + P(TT)$$

$$= p^2 + (1-p)^2$$

$$\begin{aligned} P(X=3) &= P(HTT) + P(HTH) + P(THH) + P(THT) \\ &= 1 - P(X=2) = 1 - p^2 - (1-p)^2 \end{aligned}$$

$$P(X \geq 4) = 0$$

$$\begin{aligned} E(X) &= 2 \cdot P(X=2) + 3 \cdot P(X=3) \\ &= 2(p^2 + (1-p)^2) + 3(1 - p^2 - (1-p)^2) \\ &= 3 - p^2 - (1-p)^2. \end{aligned}$$

$$\text{If } p=0, E(X) = 3 - 0 - (1-0)^2 = 2$$

$$p=1, E(X) = 3 - 1 - 0 = 2$$

$$p=\frac{1}{2}, E(X) = 3 - \frac{1}{4} - \left(\frac{1}{4}\right) = 2.5$$

#### Self test 4.4

Suppose we have  $M$  families,  $n_i$  of which have  $i$  children,  $\sum_{i=1}^r n_i = M$ .

Let  $X$  = the number of children in a randomly selected family.

Also consider, instead of picking a family at random, pick a child at random. Then  $C = \sum_{i=1}^r n_i \cdot i$ .

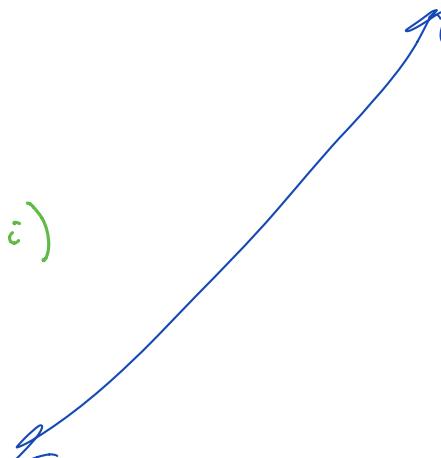
Let  $Y$  = the number of children in the family of the randomly selected child.

Goal: Show that  $E(Y) \geq E(X)$ .

$$P(X=i) = P\left(\text{choose a family with } i \text{ children}\right) = \frac{n_i}{M}.$$

$$E(X) = \sum_{i=1}^r i \cdot \frac{n_i}{M}$$

$\boxed{P(X=i)}$



$$\overline{P(Y=i)} = ?$$

$$= \frac{i \cdot n_i}{C}$$

$$E(Y) = \sum_{i=1}^r i \cdot \frac{n_i \cdot i}{C} = \sum_{i=1}^r \frac{i^2 n_i}{C}$$

$$M = \sum n_i$$

$$C = \sum i n_i$$

$$E(X) = \sum i \cdot \frac{n_i}{\sum n_i}$$

$$= \sum i \cdot \frac{n_i}{\sum n_i}$$

$$E(Y) = \sum \frac{i^2 n_i}{\sum i n_i}$$

$$= \sum i \cdot \frac{i n_i}{\sum i n_i}$$

Calculate

$$\frac{\frac{n_i}{\sum n_i}}{\frac{n_j}{\sum n_i}} = \frac{\frac{jn_j}{\sum n_i}}{\frac{n_j}{\sum n_i}} = \frac{jn_j (\sum n_i)}{n_j (\sum n_i)}$$

$$= j \frac{\sum n_i}{\sum n_i} = j \frac{M}{C}$$

$M \leq C$   
 $j \geq 1$

? Discussion of how to finish moved to campuswi...