Theory of Probability — 9/21/20

Example 5M (section 2.5)

N people.

- Mix up hats, and distribute randomly back to the people.

Question: What is the probability that no one receives their own hat?

Easiest approach Use inclusion – exclusion to compute

$$\underbrace{1 - \mathbb{P}(\text{at least one person gets their own hat})}_{\text{complementary probability.}} = \underbrace{\mathbb{P}(\text{no one gets their own hat})}_{}$$

Why is this difficult to compute on its own?

Note Every permutation of the hats has the same probability as every other permutation.

Probability that Person 1 doesn't pick their own hat $= \dfrac{N-1}{N}$

Let E_i = event that person i selects ~~their~~ their own hat

$P\left(\bigcup_{i=1}^{N} E_i\right)$ = probability that at least one person gets their own hat

Use inclusion-exclusion:

$$P\left(\bigcup_{1}^{N} E_i\right) = \sum_{i=1}^{N} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \dots + (-1)^{N+1} P(E_1 E_2 \cdots E_N).$$

Compute $P(E_{i_1} \cdots E_{i_n})$ ← probability that the $i_1, i_2, i_3, \dots i_n$th people get their own hats.

E_{i_1} is the event that person i_1 gets their hat.

Count the number of ways that people $i_1, i_2, \dots i_n$ can receive their own hats:

- n people receive their own hat
- N total people.

Each of the $N-n$ remaining hats can be distributed to the $N-n$ remaining people.

$\Rightarrow (N-n)!$ orderings

$$\Rightarrow P(E_{i_1} E_{i_2} \cdots E_{i_n}) = \frac{(N-n)!}{N!}$$

← total number of ways to distribute the hats.

Note: $P(E_1 E_3 E_7) = P(E_2 E_9 E_{15})$

How many terms are in the sum:

$$\sum_{i_1 < i_2 < \cdots < i_n} P(\underbrace{E_{i_1} E_{i_2} E_{i_3} \cdots E_{i_n}}_{n \text{ events}}) \Rightarrow \binom{N}{n} \text{ terms}$$

$$= \frac{N!}{n!(N-n)!}$$

$$\Rightarrow \sum_{i_1 < \cdots < i_n} P(E_{i_1} E_{i_2} \cdots E_{i_n}) = \binom{N}{n} \frac{(N-n)!}{N!}$$

$$= \frac{1}{n!}$$

Back to inclusion-exclusion:

$$P(\text{no one gets their hat}) = 1 - P\left(\begin{array}{c}\text{at least one person} \\ \text{gets their hat}\end{array}\right)$$

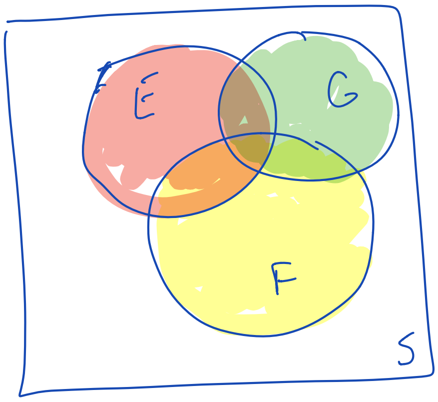
$$= 1 - P\left(\bigcup_{i=1}^{N} E_i\right)$$

$$= 1 - \left(\sum_{i=1}^{N} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \cdots + (-1)^{N+1} P(E_1 E_2 \cdots E_N)\right)$$

$$= 1 - \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \quad + (-1)^{N+1} \frac{1}{N!}\right)$$

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \approx \sum_{j=0}^{N} \frac{(-1)^j}{j!} \approx e^{-1}$$

$$= \frac{1}{e} \simeq .368$$



$$P(E \cup F)$$

$$= P(E) + P(F) - P(E \cap F)$$

$$P(E \cup F \cup G)$$

$$= P(E) + P(F) + P(G)$$

$$- P(E \cap F) - P(E \cap G) - P(F \cap G)$$

$$+ P(EFG).$$