Example 5m (section 2.5)

Mix up hats, and distribute randomly back to the people.

Question: What is the probability that no one receives their own hat?

Easiest approach: Use inclusion-exclusion to compute

\[ 1 - P(\text{at least one person gets their own hat}) = P(\text{no one gets their own hat}) \]

Complementary probability.

Why is this difficult to compute on its own?

Note: Every permutation of the hats has the same probability as every other permutation.

Probability that Person 1 doesn't pick their own hat:

\[ \frac{N-1}{N} \]
Let $E_i$ = event that person $i$ selects their own hat.

$P\left(\bigcup_{i=1}^{N} E_i\right) =$ probability that at least one person get their own hat.

Use inclusion-exclusion:

$$P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} P(E_i) - \sum_{i<i_2} P(E_i E_{i_2}) + \cdots + (-1)^{N+1} P(E_1 E_2 \cdots E_N).$$

Compute $P(E_{i_1} \cdots E_{i_n}) =$ probability that the $i_1, i_2, i_3, \ldots, i_n$ people get their own hats.

$E_{i_1}$ is the event that person $i_1$ get their hat.

Count the number of ways that people $i_1, i_2, \ldots, i_n$ can receive their own hats:

- $n$ people receive their own hat
- $N$ total people.

Each of the $N-n$ remaining hats can be distributed to the $N-n$ remaining people.

$$\Rightarrow (N-n)!$$ orderings

$$\Rightarrow P(E_{i_1} E_{i_2} \cdots E_{i_n}) = \frac{(N-n)!}{N!}$$ total number of ways to distribute the hats.
Note: \( P(E_1, E_3, E_7) = P(E_2, E_4, E_{15}) \)

How many terms are in the sum:

\[
\sum_{i_1, i_2 \cdots i_n} P(E_{i_1}, E_{i_2}, \cdots E_{i_n}) = n! \text{ terms}
\]

\[
= \frac{N!}{n! (N-n)!}
\]

\[
\Rightarrow \sum_{i_1, i_2 \cdots i_n} P(E_{i_1}, E_{i_2}, \cdots E_{i_n}) = (\frac{N}{n}) \frac{(N-n)!}{N!}
\]

\[
= \frac{1}{n!}
\]

Back to inclusion-exclusion:

\[ P(\text{no one gets their hat}) = 1 - P(\text{at least one person gets their hat}) \]

\[
= 1 - P(\bigcup_{i=1}^{N} E_i)
\]

\[
= 1 - \left( \sum_{i=1}^{N} P(E_i) - \sum_{i \neq j} P(E_i, E_j) + \cdots + (-1)^{N+1} P(E_1, E_2, \cdots E_N) \right)
\]

\[
= 1 - \left( \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \cdots + (-1)^{N+1} \frac{1}{N!} \right)
\]

\[
= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \sim \sum_{j=0}^{N} \frac{(-1)^j}{j!} \approx e^{-1}
\]

\[
= \frac{1}{e} \approx .368
\]
\[ P(E \cup F) = P(E) + P(F) - P(E \cap F) \]

\[ P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G) \]