Theory of Probability Dec 9, 2020

Stochastic Process

Idea: a sequence of random variables:

Discrete: $X_1, X_2, X_3, \ldots$

Continuous: $X(t)$

The Poisson Process

Let $N(t)$ = number of events that occur in time interval $[0, t]$.

This collection of random variables $\{N(t), t \geq 0\}$ is said to be a Poisson Process with rate λ if:

1. $N(0) = 0$
2. Number of events that occur in disjoint intervals is independent:

   If $(a,b] \cap (c,d]$ then

   $$P[N(b)-N(a) = j, \; N(d)-N(c) = k]$$
   $$= P[N(b)-N(a) = j] \; P[N(d)-N(c) = k].$$
3. The number of events in $[a,b]$ only depends on $b-a$.
4. $P[N(h) = 1] = \lambda h + o(h)$

   f is $o(h)$ if $\lim_{h \to 0} \frac{f(h)}{h} = 0$.
5. $P[N(h) \geq 2] = o(h)$.

<u>Lemma</u> For a Poisson process with rate λ,

$P[N(t) = 0] = e^{-\lambda t}$.

<u>Proof</u>: Let $P_0(t) = P[N(t) = 0]$.

$$\Rightarrow \quad P_0(t+h) = P[N(t+h) = 0]$$

$$= P[N(t) = 0, \; N(t+h) - N(t) = 0]$$

$$= P[N(t) = 0] \, P[N(t+h) - N(t) = 0]$$

$$= P_0(t) \left(1 - P[N(t+h) - N(t) \geq 1]\right)$$

$$= P_0(t) \left(1 - \lambda h + o(h)\right)$$

$$\Rightarrow \quad \frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) + \frac{o(h) P_0(t)}{h}$$

as $h \to 0$, we have in the limit

$$P_0'(t) = -\lambda P_0(t).$$

$$\Rightarrow \quad P_0(t) = C e^{-\lambda t}$$

But assumption ① of the Poisson Process was that $P_0(0) = P[N(0) = 0] = 1$

$$\Rightarrow \quad P_0(t) = e^{-\lambda t}$$

Interarrival Times

"time between events"

Let T_n = time between event n and event $n-1$

T_1 = time to the first event

How is T_n distributed?

$$P[T_1 > t] = P[N(t) = 0] = e^{-\lambda t}$$

$$\Rightarrow \quad P[T_1 \le t] = 1 - P[T_1 > t]$$

$$F_{T_1}(t) = 1 - e^{-\lambda t}$$

$$\Rightarrow \quad f_{T_1}(t) = \lambda e^{-\lambda t}$$

$$\Rightarrow \quad T_1 \sim Exp(\lambda).$$

$$P[T_2 > t] = E\big[P[T_2 > t \mid T_1]\big]$$

And now since

$$P[T_2 > t \mid T_1 = s] = P[N(t+s) - N(s) = 0 \mid T_1 = s]$$

$$= P[N(t+s) - N(s) = 0]$$

$$= P[N(t) = 0]$$

$$= e^{-\lambda t}$$

$\Rightarrow$ Value of T_1 does <u>not</u> affect the distribution of T_2. And $T_2 \sim Exp(\lambda)$.

To summarize:

<u>Proposition</u>: $T_1, T_2, T_3, \ldots$ are all $\text{Exp}(\lambda)$ random variables.

<u>Waiting Times</u> (Arrival times)

S_n = arrival time of the n^{th} event

$\neq T_n$

$$S_n = \sum_{i=1}^{n} T_i \quad \sim \text{Exp}(\lambda).$$

$\Rightarrow$ From section 5.6, we know that

$$S_n \sim \text{Gamma}(n, \lambda).$$

$$\Rightarrow \quad f_{S_n}(s) = \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!}$$

$$= \lambda^n e^{-\lambda s} \frac{s^{n-1}}{(n-1)!} \quad \text{for } s \geq 0.$$

<u>Theorem</u>: For a Poisson process with rate λ,

$$P[N(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

<u>Idea of Proof</u>: $P[N(t)=n] = P[N(t) \geq n] - P[N(t) \geq n+1]$

$$= P[S_n \leq t] - P[S_{n+1} \leq t].$$

Use this form to derive distribution.