Strong Law of Large Numbers (SLLN)

Let $X_i, X_j \ldots$ be a sequence of IID random variables, with $E[X_i] = \mu < \infty$. Then

$$P\left[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i = \mu \right] = 1.$$

Weak Law of Large Numbers (WLLN)

For any $\epsilon > 0$, \( \lim_{n \to \infty} P\left[ \left| \frac{1}{n} \sum_{i=1}^{n} X_i - \mu \right| > \epsilon \right] = 0. \)

Differences: Additional assumptions in the proof in the text:

- WLLN: $\text{Var}[X_i] = \sigma^2 < \infty$
- SLLN: $E[X_i^4] < \infty$.

Furthermore:

$\text{SLLN} \Rightarrow \text{WLLN}$.

Additional Inequalities

We may be interested in estimating

$$P\left[ X - \mu > a \right]$$

when only $E[X_i] = \mu$ and $\text{Var}[X] = \sigma^2$ are known, (for $a > 0$)
Trivially, since $X - \mu \geq a \Rightarrow |X - \mu| \leq a$, we can immediately apply Chebyshev's Inequality:

$$P[ X - \mu \geq a ] \leq P[ |X - \mu| \geq a ] \leq \frac{\sigma^2}{a^2} \text{ for } a > 0.$$ 

**Proposition One-sided Chebyshev Inequality:**

If $E[X] = 0$, $P[ X \geq a ] \leq \frac{\sigma^2}{\sigma^2 + a^2} \leq \frac{\sigma^2}{a^2}$.

**Proof:** Let $b > 0$ and note that $X \geq a$ is equivalent to $X + b \geq a + b$,

$$P[ X \geq a ] = P[ X + b \geq a + b ]$$

$$\leq P[ (X + b)^2 \geq (a + b)^2 ]$$

$$\leq \frac{E[ (X + b)^2 ]}{(a + b)^2} = \frac{\sigma^2 + b^2}{(a + b)^2}$$

Set $b = \frac{\sigma^2}{a}$ (which minimizes $\frac{\sigma^2 + b^2}{(a + b)^2}$)

$$\Rightarrow \quad P[ X \geq a ] \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$ 

**Jensen's Inequality**

If $f$ is a convex function, i.e., $f''(x) > 0$ for all $x$, then $E[ f(X) ] \geq f( E[X] )$,

assuming $E[ f(X) ]$, $E[X]$ exist and are finite.
Proof:

Write \( f(x) = f(\mu) + f'(\mu)(x-\mu) + \frac{f''(s)(x-\mu)^2}{2}, \quad s \in [\mu, x] \)

Since \( f''(s) \geq 0 \),

\[ f(x) \geq f(\mu) + f'(\mu)(x-\mu) \]

\[ \Rightarrow f(x) > f(\mu) + f'(\mu)(x-\mu) \]

\[ \Rightarrow E[f(x)] > E[f(\mu) + f'(\mu)(X-\mu)] \]

\[ = f(\mu) + f'(\mu)E[X-\mu] \]

\[ = f(\mu) + f'(\mu)E[X] \]

\[ = f(E[X]), \quad \checkmark \]

Idea:

The line \( y = f(\mu) + f'(\mu)(x-\mu) \) is always below the curve \( y = f(x) \)

\[ \Rightarrow f(x) > f(\mu) + f'(\mu)(x-\mu). \]