Theory of Probability

Dec 2

Weak Law of Large Numbers (WLLN)

Let $X_1, X_2, \dots$ be IID random variables with $E[X_i] = \mu < \infty$ and $Var[X_i] = \sigma^2 < \infty$.

Then for any $\epsilon > 0$,

$$P\left[\left| \frac{1}{n} \sum_{i=1}^{n} X_i - \mu \right| \geq \epsilon \right] \to 0 \quad \text{as } n \to \infty.$$

To prove WLLN, we first need some inequalities:

Markov Inequality

If $X \geq 0$ is a random variable, then for any $a > 0$,

$$P[X \geq a] \leq \frac{E[X]}{a}.$$

Proof:

$$a P[X \geq a] = \int_a^{\infty} a f(x)\, dx$$

$$\leq \int_a^{\infty} x f(x)\, dx$$

$$\leq \int_0^{\infty} x f(x)\, dx = E[X]$$

$$\Rightarrow \quad a P[X \geq a] \leq E[X]$$

$$\Rightarrow \quad P[X \geq a] \leq \frac{E[X]}{a}.$$

Chebyshev's Inequality

Let X be a random variable with $E[X] = \mu < \infty$ and $Var[X] = \sigma^2 < \infty$.
Then for any $k > 0$,

$$P[\,|X-\mu| \geq k\,] \leq \frac{\sigma^2}{k^2}$$

Proof: By Markov's Inequality,

$$\underbrace{P[\,(X-\mu)^2 \geq k^2\,]}\; \leq \frac{E[(X-\mu)^2]}{k^2}$$

$$\Leftrightarrow P[\,|X-\mu| \geq k\,] \leq \frac{\sigma^2}{k^2}.$$

Proof of WLLN:

Note that $E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \mu$

and $Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{\sigma^2}{n}$.

Using Chebyshev's Inequality, we have that

$$P\left[\,\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\,\right] \leq \sigma^2/n \big/ \epsilon^2 = \frac{\sigma^2}{n\epsilon^2}$$

$$\lim_{n\to\infty} P\left[\,\left|\frac{1}{n}\sum X_i - \mu\right| \geq \epsilon\,\right] = 0 \quad \checkmark$$

The Central Limit Theorem

Let $X_1, X_2, \ldots$ be IID with $E[X_i] = \mu < \infty$

$Var[X_i] = \sigma^2 < \infty$.

Then

$$\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\,\sigma} \rightsquigarrow N(0,1)$$

as $n \to \infty$.

I.e.,

$$\lim_{n\to\infty} P\left[\frac{\frac{1}{n}\sum_{1}^{n} X_i - \mu}{\sigma/\sqrt{n}} \le a\right] = \Phi(a).$$

This theorem can be extended to many more general cases where X_i are independent, and

$$E[X_i] = \mu_i < \infty$$

and $Var[X_i] = \sigma_i^2 < \infty$

and if (a) $P[\,|X_i| < M\,] = 1$ (bounded)

(b) $\sum_{i=1}^{\infty} \sigma_i^2 = \infty$

Then $$\lim_{n\to\infty} P\left[\frac{\sum_{i=1}^{n}(X_i - \mu_i)}{\sqrt{\sum_{i=1}^{n}\sigma_i^2}} \le a\right] = \Phi(a).$$