Moment Generating Function

$$M(t) = E[e^{tX}]$$
$$= \int e^{tx} f(x)\, dx$$

Similar to the Laplace Transform of f

(the function M for $t \neq 0$, might not exist.)

Usefulness:

$$M(0) = E[e^0] = E[1] = 1$$

$$M'(t) = \frac{d}{dt} \int e^{tx} f(x)\, dx$$
$$= \int \frac{d}{dt} e^{tx} f(x)\, dx$$
$$= \int x e^{tx} f(x)\, dx$$
$$\Rightarrow M'(0) = \int x f(x)\, dx = E[X].$$
$$M''(0) = E[X^2]$$
$$\Rightarrow M^{(n)}(0) = E[X^n].$$

Ex: Std. Normal

$$M(t) = E[e^{tZ}] = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$
$$= e^{t^2/2}$$

$$M'(t) = \frac{2t}{2} e^{t^2/2} = t e^{t^2/2}$$

$$M'(0) = 0 = \text{Mean } (E[Z])$$

$$M''(t) = e^{t^2/2} + t^2 e^{t^2/2}$$

$$M''(0) = 1 = Var[Z] = E[Z^2].$$

For $X \sim N(\mu, \sigma^2)$

$$X = \mu + \sigma Z$$

$$M_X(t) = E[e^{tX}] = E[e^{t(\mu + \sigma Z)}]$$

$$= e^{t\mu} E[e^{t\sigma Z}]$$

$$= e^{t\mu} e^{t^2\sigma^2/2}$$

$$= e^{t\mu + t^2\sigma^2/2}$$

If X, Y are independent, then

$$M(t) = E[e^{t(X+Y)}] = E[e^{tX} e^{tY}]$$

$$= E(e^{tX}) E[e^{tY}]$$

$$= M_X(t) M_Y(t)$$

Lastly: If $M(t) < \infty$ and exists in a neighborhood of $t=0$, then this uniquely defines the probability distribution (i.e. the PDF).

Joint MGF

If $X_1, \ldots, X_n$ have joint pdf $f(x_1, \ldots, x_n)$, then

$$M(t_1, t_2, \ldots, t_n) = \int \cdots \int e^{t_1 x_1 + t_2 x_2 + \cdots + t_n x_n} f(x_1, \ldots, x_n)\, d\vec{x}.$$

$$= E\left[e^{t_1 X_1 + t_2 X_2 + \cdots + t_n X_n}\right]$$

$$M_{X_i}(t) = M(0, \ldots, 0, t_i, 0 \ldots 0)$$

Additional Normal R.V. Properties

Let $Z_1, \ldots, Z_n$ be IID standard normal random variables. For constants μ_i, a_{ij} $i = 1 \ldots m$, $j = 1, \ldots, n$, define:

$$X_1 = a_{11} Z_1 + a_{12} Z_2 + \cdots + a_{1n} Z_n + \mu_1$$

$$\vdots$$

$$X_m = a_{m1} Z_1 + a_{m2} Z_2 + \cdots + a_{mn} Z_n + \mu_m.$$

$$\vec{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} = A\vec{Z} + \vec{\mu} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{m1} & & a_{mn} \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}$$

$\Rightarrow$ $E[X_i] = \mu_i$

$$Var[X_i] = Var\left[\sum_{j=1}^{n} a_{ij} Z_j\right]$$

$$= \sum_{j=1}^{n} a_{ij}^2.$$

$$\mathrm{Cov}(X_i, X_j) = \mathrm{Cov}\left(\sum_{k=1}^{n} a_{ik} Z_k, \sum_{k'=1}^{n} a_{jk'} Z_{k'}\right)$$

$$= \sum_{k,k'} a_{ik} a_{jk'} \mathrm{Cov}(Z_k, Z_{k'})$$

$$C_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk} \quad \Big\} \text{ Inner product of } \vec{a}_i \text{ with } \vec{a}_j .$$

<u>The Joint MGF for $X_1 \dots X_m$</u>

$$M(t_1, \dots, t_m) = e^{\sum_i^m t_i \mu_i + \frac{1}{2} \sum_{ij} t_i t_j C_{ij}}$$

$\Rightarrow$ Since M determines the PDF, and M depends only on μ_i, C_{ij}, then the joint PDF must <u>only</u> depend on μ_i, C_{ij}.

$$f(x_1, \dots, x_m) = \frac{1}{(2\pi)^{m/2} \sqrt{|C|}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T C^{-1} (\vec{x}-\vec{\mu})} .$$

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \cdots & C_{1m} \\ \vdots & \ddots & & \\ C_{m1} & & & C_{mm} \end{pmatrix} .$$