

Conditional Distributions

Goal: Given  $X, Y$  random variables with joint pdf  $f(x,y)$  and marginal pdfs.  $f_x, f_y$ , what is the pdf of the random variable

$$Z = \underbrace{X}_{\text{"}X \text{ given } Y\text{"}} \mid Y=y$$

First: Discrete Case

Recall for events :

$$P[E|F] = \frac{P[EF]}{P[F]}$$

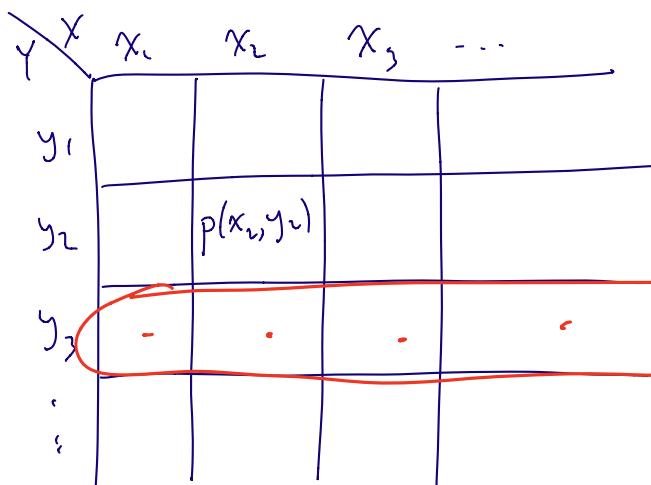
Define the conditional probability mass function  
so that

$$\begin{aligned} p_{X|Y}(x|y) &= P[X=x \mid Y=y] \\ &= \frac{P[X=x, Y=y]}{P[Y=y]} \\ &= \frac{p(x,y)}{p_Y(y)} \quad \begin{array}{l} \text{joint pdf function} \\ \text{marginal mass function.} \end{array} \\ &\quad \text{(assuming } p_Y(y) > 0\text{).} \end{aligned}$$

Conditional Distribution function is then

$$F_{X|Y}(x|y) = P[X \leq x | Y=y]$$

$$= \sum_{a \leq x} P[X=a | Y=y]$$



Grid of probabilities

Fix  $Y=y_3$ ,

scale this row by  
 $\frac{1}{P_Y(y_3)}$  so that

$$\sum_i P_{X|Y}(x_i | y_3) = 1.$$

If  $X, Y$  are independent random variables,

then  $p(x,y) = p_X(x) p_Y(y)$ .

$$\Rightarrow P_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$$

$$= \frac{p_X(x) p_Y(y)}{p_Y(y)}$$

$$= p_X(x).$$

### Continuous Case

$$\text{Defini} \quad f_{x|y}(x|y) = \frac{f(x,y)}{f_y(y)}$$

Motivation : multiply each side by  $dx = \frac{dx dy}{dy}$

$$\Rightarrow f_{x|y}(x|y) dx = \frac{f(x,y) dx dy}{f_y(y) dy}$$

$$\approx \frac{P[X \in (x, x+dx), Y \in (y, y+dy)]}{P[Y \in (y, y+dy)]}.$$

$$\Rightarrow P[X \in A | Y=y] = \int_A f_{x|y}(x|y) dx$$

$$\text{likewise } P[X \in A | Y \in B] = \int_A \int_B f_{x|y}(x|y) dy dx.$$

Distribution function :

$$A = (-\infty, x)$$

$$\Rightarrow F_{x|y}(x|y) = P[X \leq x | Y=y]$$

$$= \int_{-\infty}^x f_{x|y}(u|y) du.$$

If  $X, Y$  are independent , then

$$f_{x|y}(x|y) = f_x(x).$$